# A PENTAGONAL CRYSTAL, THE GOLDEN SECTION, ALCOVE PACKING AND APERIODIC TILINGS.

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## Abstract

A Lie theoretic interpretation is given to a pattern with five-fold symmetry occurring in aperiodic Penrose tiling based on isosceles triangles with length ratios equal to the Golden Section. Specifically a $B(\infty)$ crystal based on that of Kashiwara is constructed exhibiting this five-fold symmetry. It is shown that it can be represented as a Kashiwara $B(\infty)$ crystal in type $A_4$. Similar crystals with $(2n+1)$-fold symmetry are represented as Kashiwara crystals in type $A_{2n}$. The weight diagrams of the latter inspire higher aperiodic tiling. In another approach alcove packing is seen to give aperiodic tiling in type $A_4$. Finally $2m$-fold symmetry is related to type $B_m$.

## 1. INTRODUCTION

1.1.   This work arose as an attempt to explicitly describe and more deeply understand the $B(\infty)$ crystal introduced by Kashiwara [8, Sections 0,4]. Let us first recall the context in which it is described.

1.2.   Let $C$ be a Cartan matrix in the sense used to define Kac-Moody algebras. More precisely $C$ is a square matrix of finite (or even countable size) with diagonal entries equal to 2 and non-positive integer off-diagonal entries. In the Kashiwara theory one needs $C$ to be symmetrizable in order to introduce the associated quantized enveloping algebra from which the (purely combinatorial) properties of $B(\infty)$ are deduced. However using the Littelmann path model [12] it is possible (1.6) to weaken this to the requirement that the $ij^{th}$ entry of $C$ be non-zero if and only if

the $ji^{th}$ entry be non-zero. This is of course exactly the condition under which the Kac-Moody algebra $\mathfrak{g}$ associated to $C$ is defined [7].

1.3.  The $B(\infty)$ crystal is a purely combinatorial object which can viewed as providing a basis (the crystal basis) of the algebra of functions on the open Bruhat cell defined by $\mathfrak{g}$. The latter is of course a polynomial algebra (on possibly infinite many variables) though this is very far from obvious from the combinatorial description of $B(\infty)$. Indeed one only knows its formal character to have the expected product form by means which are particularly roundabout, especially in the non-symmetrizable case. Moreover the combinatorial complexity of $B(\infty)$ is essential in that it leads in a simple manner to a crystal basis for each highest weight integrable module.

1.4.  The $B(\infty)$ crystal is specified entirely in terms of the Cartan matrix $C$ by the following very simple procedure. First as in Kac [7, Chapter 2] one realizes $C$ through a vector space $\mathfrak{h}$ (eventually the Cartan subalgebra of $\mathfrak{g}$), a set of simple coroots in $\mathfrak{h}$ and a set $\Delta$ of simple roots in $\mathfrak{h}^*$ so that the entries of C are given by evaluation of coroots on roots. Moreover using these roots and coroots we may define the Weyl group $W$ in the usual way.

1.5.  To each simple root $\alpha$, Kashiwara [9, Example 1.2.4] introduced an "elementary crystal" $B_\alpha$ whose elements can be viewed simply as non-positive multiples of $\alpha$, so then $B_\alpha$ is identified with $\mathbb{N}$. (We have changed Kashiwara's definition slightly (see [2, 12.3]). Now fix any countable sequence $J$ of simple roots (indexed by the positive integers) with the property that every simple root occurs infinitely many times and take the subset $B_J$ of the corresponding tensor product of the elementary crystals having only finitely many non-zero entries. Of course $B_J$ has a distinguished element, denoted $b_\infty$, in which all entries are equal to zero. One views the elements of $B_J$ as forming the vertices of a graph (the crystal graph). On $B_J$ one defines a Kashiwara function $r$ with entries in $\mathbb{Z}$, given by a very simple formula involving just the entries of the Cartan matrix $C$. Its role is to describe the edges of the crystal graph which are labelled by the simple roots. Indeed inequalities between the values of $r$ on a given vertex $b$ decide the neighbours of $b$. Finally $B_J(\infty)$ is defined to be the connected component of $B_J$ containing $b_\infty$ .

1.6.  A deep and important result of Kashiwara is that as a graph $B_J(\infty)$ is independent of $J$. Kashiwara's result is obtained via the quantized enveloping algebra using a $q \to 0$ limit. (Lusztig [13] has a different version of this limit and the resulting combinatorics.) It requires $C$ to be symmetrizable; but this condition can be dropped through a purely combinatorial proof using the Littelmann path model [2, 11.16, 15.11, 16.10].

1.7.    One can ask if it is possible to describe $B_J(\infty)$ explicitly as a subset of $B_J$. Of course this should involve the Cartan matrix which is in effect the only ingredient in the determination of $B_J(\infty)$. However this occurs in an extremely complicated fashion and it is even rather difficult to establish general properties of the embedding [10, 15]. Nevertheless it was noted by Kashiwara [9, Prop. 2.2.3] that the rank 2 case is manageable. Here we note that this solution involves the Chebyshev polynomials with argument being the square root of the product of the two off-diagonal entries of $C$. In truth these are not quite the Chebyshev polynomials as customarily defined, however the difference will probably not bother most readers. We refer the fastidious to 2.2.

1.8.    The square of the largest zero of the $n^{th}$ Chebyshev polynomial is $< 4$ and tends to this value as n tends to infinity. In particular the largest non-negative integer values must be $0, 1, 2, 3$ and these occur as the squares of the largest zeros for just the second, third, fourth and sixth Chebyshev polynomial. Moreover such a zero results in a cut-off in the description of $B_J(\infty)$ which as a consequence lies in a finite Cartesian product of elementary crystals.

1.9.    One can ask if the squares of the largest zeros of the remaining Chebyshev polynomials leads to a similar cut-off in the description of $B_J(\infty)$. The first interesting case is the fifth Chebyshev polynomial whose largest zero is the Golden Section $g$. Of course since $g$ is not an integer or even rational one needs to modify the definition of $B_J(\infty)$ for the construction to make any sense.

1.10.    In a similar vein one does not need the Cartan matrix to have integer off-diagonal entries in order to define the Weyl group. That the resulting group be finite (in rank two) similarly involves the largest zeros of the Chebyshev polynomials. In this fashion the $n^{th}$ Chebyshev polynomial gives a Weyl group isomorphic to the dihedral group of order $2n$. Returning to the case of $n = 5$ this leads to a "root system" having 5 positive roots which matches with the expectation that $B_J(\infty)$ embeds in a five-fold tensor product.

1.11.    The fact that the Golden Section $g$ is irrational and satisfies a quadratic equation means that we may retain a purely integer set-up in the definition of $B_J$ by adding collinear roots of relative length $g$. This gives in all twenty non-zero roots. A link with mathematics of the ancient world is that the resulting root system can be described as the projection of the vertices of a dodecahedron onto the plane defined by one of its faces - see Figure 1.

1.12. A further justification for introducing pairs of collinear roots comes from the nature of the Weyl group itself, which has two generators and isomorphic to $\mathbb{Z}_5 \ltimes \mathbb{Z}_2$, that is the dihedral group of order 10. Because $g$ is not rational but satisfies a quadratic equation there is a natural decomposition of each of the two simple reflections into two commuting involutions giving a larger group on four generators, which we call the augmented Weyl group $W^a$, see 3.9. To our surprise this larger group (which acts just $\mathbb{Z}$ linearly and not isometrically) leaves the enlarged root system invariant. Having said this it is no surprise that this larger group is just the permutation group on 5 elements and obtained as the symmetry group of the dodecahedron via the projection described in 1.11.

1.13. In view of the above rather pleasing geometric interpretation, it became a seemingly worthwhile challenge to indeed construct and describe explicitly a "pentagonal crystal", in the sense of the Kashiwara $B_J(\infty)$, proving its independence on $J$ and computing its formal character.

1.14. The above program was carried out, not without some difficulty. Indeed the inequalities which describe $B_J(\infty)$ as a subset of $B_J$ are significantly more complicated than a naive interpretation of our previous inequalities from the rank 2 case would suggest.

1.15. The reader may spare himself the detailed verification of the assertions alluded to in 1.14 since it becomes apparent that our pentagonal crystal is a realization of a Kashiwara $B_J(\infty)$ crystal in type $A_4$ and $W^a$ the corresponding Weyl group $W(A_4)$ in type $A_4$. Thus existence and independence of $J$ may be proved by exhibiting this isomorphism. Nevertheless our computation was not entirely in vain. Indeed for some special choices of J the description of $B_J(\infty)$ in type $A$ is particularly simple [15]. However these are not the choices required here and for them the resulting description is far more complicated.

1.16. In view of 1.15 it is natural to ask if the remaining largest zeros of Chebyshev polynomials for $n \geqslant 1$ lead to crystals with a similar interpretation. Indeed the cases of the third and fifth Chebyshev polynomials are just special cases of the crystals obtained from the $(2n+1)^{th}$ Chebyshev polynomial. It turns out that the $(2n+1)^{th}$ Chebyshev polynomial factorizes into a pair of polynomials (related by replacing the argument by its negative). These are irreducible over $\mathbb{Q}$, if and only if $(2n+1)$ is prime. This leads to n collinear roots replacing each positive (or negative) root leading to a total of 2n(2n+1) roots which just happens to be the number of roots in type $A_{2n}$. In 7.7 we exhibit the required isomorphism with $B_J(\infty)$ for a particular choice of $J$. However one should note that there is an important distinction with the pentagonal case if $(2n+1)$ is not prime. In particular we show that the augmented Weyl group $W^a$ is just the Weyl group $W(A_{2n})$ in type $A_{2n}$. In Section 10, we consider

the even case for which $W \cong \mathbb{Z}_{2m} \ltimes \mathbb{Z}_2$. We show that then $W^a$ is isomorphic to the Weyl group $W(B_m)$ in type $B_m$. One would clearly like to extend this connection for all finite reflection groups $W$, that is to say construct $W^a$ and show it to be isomorphic to the Weyl group of a root system.

1.17. We remark that although our pentagonal root system is just an appropriate orthogonal projection of a dodecahedron, the latter cannot be obtained from a similar orthogonal projection of the root system of type $A_4$. Yet the relation between the Coxeter groups of type $A_4$ and the pentagonal system may be thought to be an extension of the traditional one obtained by say embedding a root system of type $G_2$ into one of type $B_3$ via a seven dimensional representation of the Lie algebra of the former.

1.18. In Section 8 we consider Penrose aperiodic tiling based on the two isosceles triangles (the Golden Pair 8.7) whose unequal side lengths ratios is the Golden Section. We view these triangles as being obtained by triangularization of the regular pentagon. We show (Theorem 8.14) that the triangles obtained by the regular $n$-gon lead to a higher aperiodic tiling, though this is a totally elementary result having no Lie theory content. In Section 9 we suggest that such tilings can be thought of as a consequence of alcove packing in the Cartan subalgebras whose associated Weyl group is the augmented Weyl group. An explicit construction is given in the pentagonal case, 9.10, 9.11. Aperiodicity (which we view as the possibility to obtain arbitrary many tilings) corresponds to using different sequences of reflections in the affine Weyl group. However for the moment our construction does not give all possible tilings.

## 2. Root systems

Throughout the base field will be assumed to be the real numbers $\mathbb{R}$.

2.1. Let $\mathfrak{h}$ be a vector space and $I := \{1, 2, \ldots, n\}$. Define a root pair $(\pi^\vee, \pi)$ to consist of a set $\pi^\vee = \{\alpha_i^\vee \mid i \in I\}$ of linearly independent elements (called simple coroots) of $\mathfrak{h}$ and a set $\pi = \{\alpha_i \mid i \in I\}$ of linearly independent elements (called simple roots) of $\mathfrak{h}^*$ such that $\alpha_i^\vee(\alpha_i) = 2$, for all $i \in I$. For all $i \in I$, define the simple reflection $s_i \in Aut\ \mathfrak{h}^*$ by

$$s_i \lambda = \lambda - \alpha_i^\vee(\lambda)\alpha_i,$$

and let $W$ be the group they generate. It will be assumed that $\alpha_i^\vee(\alpha_j) = 0$, if and only if $\alpha_j^\vee(\alpha_i) = 0$. The matrix with entries $\alpha_i^\vee(\alpha_j)$ will be called the Cartan matrix. For the moment we shall only assume that its off-diagonal entries are non-positive reals.

2.2.   Take $n = 2$ in 2.1. Set $\alpha = \alpha_1, \beta = \alpha_2, s_\alpha = s_1, s_\beta = s_2$ . Since we do not mind introducing possibly superfluous square roots we shall symmetrize the Cartan matrix so that its off-diagonal entries are both equal to $-x$. Observe that

$$s_\alpha s_\beta \alpha = (x^2 - 1)\alpha + x\beta, \ s_\alpha s_\beta \beta = -\beta - x\alpha.$$

Thus if we define functions $R_n(x), S_n(x)$ by

$$(s_\alpha s_\beta)^n \alpha = R_n(x)\alpha + S_n(x)\beta,$$

we find that $R_n, S_n$ are defined by the recurrence relations

$$S_{n+1} = xR_n - S_n, R_{n+1} = (x^2 - 1)R_n - xS_n = xS_{n+1} - R_n : n > 0,$$

with the initial conditions $S_0 = 0, R_0 = 1$.

These relations are exactly satisfied by setting

$$R_n = P_{2n}, S_n = P_{2n-1},$$

where the $P_n$ satisfy the recurrence relation

$$P_{n+1} = xP_n - P_{n-1}, \forall n \geq 0,$$

with the initial condition $P_{-1} = 0, P_0 = 1$. One may check that

$$(\sin \theta)P_n(2\cos\theta) = \sin(n+1)\theta, \forall n \in \mathbb{N}. \tag{$*$}$$

A few examples are given below

$$P_0 = 1, P_1 = x, P_2 = x^2 - 1, P_3 = x(x^2 - 2), P_4 = x^4 - 3x^2 + 1, P_5 = x(x^2 - 3)(x^2 - 1),$$

$$P_6 = x^6 - 5x^4 + 6x^2 - 1, P_7 = x(x^2 - 2)(x^4 - 4x^2 + 2), P_8 = (x^2 - 1)(x^6 - 6x^4 + 9x^2 - 1).$$

Set $\theta = \pi/n + 1$. Then by $(*)$ the $2\cos t\theta : t \in 1, 2, \ldots, n$, form the set of zeros of the degree $n$ polynomial $P_n$. Thus these zeros are pairwise distinct and real with $x := 2\cos\pi/(n+1)$ being the largest. Moreover $x$ is just the third length of the isosceles triangle with equal side lengths 1 and equal angles $\theta$. Finally $(\pi - \theta)$ is just the angle between the vectors $\alpha$ and $\beta$ in Euclidean two-space. We will refer to $P_n$ as the $(n+1)^{th}$ Chebyshev polynomial. To be precise the "true" Chebyshev polynomials $P_n^c$ only coincide with the $P_n$ for $n = 0, 1$. Otherwise they are defined by the recurrence relation

$$P_{n+1}^c = 2xP_n^c - P_{n-1}^c, \forall n \geq 0.$$

One may check that

$$P_{n+1}^c(x) = P_{n+1}(2x) - xP_n(2x), \quad P_n^c(\cos\theta) = \cos n\theta, \forall n \in \mathbb{N}.$$

A few examples are given below

$$P_0^c = 1, P_1 = x, P_2^c = 2x^2 - 1, P_3^c = 4x^3 - 3x, P_4^c = 8x^4 - 8x^2 + 1.$$

2.3. In the above situation the Weyl group $W = < s_\alpha, s_\beta >$ is finite if and only if $(s_\alpha s_\beta)^n = 1$, for some $n \geq 2$. Assume that n is the least integer with this property. Observe that $n = 2$, exactly when $x = 0$. In general $s_\alpha s_\beta$ is a rotation by $2\pi/n$. Thus if $n$ is even, say $n = 2m$, then $(s_\alpha s_\beta)^m$ is a rotation by $\pi$ and so acts by $-1$. This is satisfied by the vanishing of $S_m$. In $n$ is odd, say $n = 2m + 1$, then

$$(s_\alpha s_\beta)^m \alpha = (s_\beta s_\alpha)^{m+1} \alpha = -s_\beta (s_\alpha s_\beta)^m \alpha.$$

Consequently $(s_\alpha s_\beta)^m \alpha$ is a multiple of $\beta$ and similarly $(s_\beta s_\alpha)^m \beta$ is a multiple of $\alpha$. These conditions are satisfied by the vanishing of $R_m$. In all cases $W \cong \mathbb{Z}_n \ltimes \mathbb{Z}_2$, that is the dihedral group of order $2n$.

## 3. Crystals

3.1. Adopt the conventions of 2.1. A crystal $B$ is a countable set whose elements are viewed as vertices of a graph (the crystal graph, also denoted by $B$). The edges of $B$ are labelled by elements of $\alpha$ with the following two conditions imposed.

1) *Removing all edges except one results in a disjoint union of linear graphs.*

2) *There is a weight function $wt : B \to \mathfrak{h}^*$ with the property that $wt\, b - wt\, b' \in \{\pm\alpha\}$, if $b, b'$ are joined by an edge labelled by $\alpha$.*

By 1), 2) we may define maps $e_\alpha, f_\alpha : B \to B \cup \{0\}$ by $e_\alpha b' = b$ if and only if $f_\alpha b = b'$, whenever $b, b'$ are non-zero, with $e_\alpha$ (resp. $f_\alpha$) increasing (resp. decreasing) weight by $\alpha$.

Any subset $B'$ of $B$ inherits a crystal structure by deleting all edges joining elements of $B'$ to $B \setminus B'$. We say that $B'$ is a strict subcrystal of $B$ if no edges need be deleted, that is as a graph, $B'$ is a component of $B$.

Let $\mathcal{E}$ (resp. $\mathcal{F}$) denote the monoid generated by the $e_\alpha$ (resp. $f_\alpha$):$\alpha \in \pi$. A crystal is said to be of highest weight $\lambda \in \mathfrak{h}^*$ if there exists $b \in B$ of weight $\lambda$ satisfying $\mathcal{E}b = 0$ and $\mathcal{F}b = B$.

3.2. A crucial component of crystal theory is tensor structure. As a set the tensor product $B \otimes B'$ of crystals $B, B'$ is simply the Cartesian product where the weight function satisfies $wt(b \otimes b') = wt\, b + wt\, b'$. In order to assign edges to the required graph Kashiwara [9, Definition 1.2.1] introduced auxillary functions $\varepsilon_\alpha : B \to \mathbb{Z} \cup \{-\infty\} : \alpha \in \pi$, with the property that $\varepsilon_\alpha(e_\alpha b) = \varepsilon_\alpha(b) - 1$, whenever $e_\alpha b \neq 0$.

Let us pass immediately to a multiple tensor product $B_n \otimes B_{n-1} \otimes \ldots \otimes B_1$. On the corresponding Cartesian product we shall define the edges through the Kashiwara

function given on the element $b_n \times b_{n-1} \times \ldots \times b_1 : b_i \in B_i$ , by

$$r_\alpha^k(b) = \varepsilon_\alpha(b_k) - \sum_{j=k+1}^{n} \alpha^\vee(wt\ b_j).$$

3.3.   Let us assume for the moment that the Cartan matrix is classical, namely has non-positive integer diagonal entries. Set

$$\varepsilon_\alpha(b) = \max_k r_\alpha^k(b),\ \ g_\alpha(b) = \max_k\{\varepsilon_\alpha(b_k) = \varepsilon_\alpha(b)\},\ \ d_\alpha(b) = \min_k\{\varepsilon_\alpha(b_k) = \varepsilon_\alpha(b)\}.$$

The Kashiwara tensor product rule [9, Lemma 1.3.6] is given by

   i) $e_\alpha b = b_n \times \ldots \times e_\alpha b_t \times \ldots \times b_t$, where $t = g_\alpha(b)$,

   ii) $f_\alpha b = b_n \times \ldots \times f_\alpha b_t \times \ldots \times b_t$, where $t = d_\alpha(b)$.

One checks that this gives the Cartesian product a crystal structure. It is manifestly associative, but it is not commutative. When (i) (resp. (ii)) holds we say that $e_\alpha$ (resp. $f_\alpha$) enters $b$ at the $t^{th}$ place.

3.4.   When one begins to tamper with the entries of $C$ taking them to be arbitrary reals, some of the required properties may fail, particularly that noted in the last part of 3.1. This is already the case when the diagonal elements are replaced by non-positive integers. For this case a cure has been given by Jeong, Kang, Kashiwara and Shin [6]. Non-integer (real) entries are also problematic. Indeed suppose that $f_\alpha b \neq 0$ and set $t = d_\alpha(b)$. This means that $r_\alpha^s(b) < r_\alpha^t(b)$, for all $s < t$. On the other hand $r_\alpha^t(f_\alpha b) = 1 + r_\alpha^t(b)$, whilst $r_\alpha^s(f_\alpha b) = \alpha^\vee(\alpha) + r_\alpha^s(b) = 2 + r_\alpha^s(b)$, for $s < t$. Thus to obtain $e_\alpha f_\alpha b = b$ , we require that $r_\alpha^s(b) < r_\alpha^t(b)$ to imply $r_\alpha^s(b) \leqq r_\alpha^t(b) - 1$, which is true if the Kashiwara functions take integer values, but may fail otherwise.

3.5.   Non-integer values of the Kashiwara function are inevitable if $C$ has non-integer entries. Our remedy is to assume that the entries of $C$ take values in a ring which is free finitely generated $\mathbb{Z}$ module $M = \mathbb{Z}g_1 + \mathbb{Z}g_2 + \ldots + \mathbb{Z}g_s$, with $g_1 = 1$. If the Cartan matrix $C$ has real entries, then $M$ one might expect to take M to be a subring of $\mathbb{R}$. However it is rather more convenient to allow $M$ to have zero divisors. Given $m \in M$, let $m_i$ denote its $i^{th}$ component (in which we always omit the multiple of $g_i$). Define

$$P = \{\lambda \in \mathfrak{h}^*|\alpha^\vee(\lambda) \in M, \forall \alpha \in \pi\}, P^+ = \{\lambda \in P|\alpha^\vee(\lambda)_i \geq 0, \forall \alpha \in \pi, \forall i \in \{1, 2, \ldots, s\}\}.$$

Assume that $wt$ takes values in $P$. Further assume that the $\varepsilon_\alpha$ take values in $M$. Consequently the Kashiwara function will also take values in $M$. Let $\mathbf{g}_i$ denote the element in M with $g_i$ in the $i^{th}$ entry and zeros elsewhere.

Following the above we extend the notion of a crystal in the following obvious fashion. Define for all $\alpha \in \pi$ and $i \in \{1, 2, \ldots, s\}$ maps $e_{\alpha,i}, f_{\alpha,i} : B \to B \cup \{0\}$ satisfying $e_{\alpha,i}b' = b$, if and only if $f_{\alpha,i}b = b'$ whenever $b, b'$ are non-zero with $e_{\alpha,i}$ (resp. $f_{\alpha,i}$) increasing (resp. decreasing) weight by $\mathbf{g}_i\alpha$ and decreasing $\varepsilon_\alpha$(resp. increasing) by $\mathbf{g}_i$.

Continue to define the Kashiwara functions as in 3.2. Its component in the $i^{th}$ factor will be an integer and as in 3.3 provides the rule for the insertion of the $e_{\alpha,i}, f_{\alpha,i}$, in a multiple product.

Notice that our algorithm is being applied to each simple root at a time. Thus we can allow ourselves the flexibility of using different $\mathbb{Z}$ bases for $M$. In some cases it is even convenient to allow $M$ itself to depend on the simple root in question (see Section 10).

3.6. Return to the case of a classical Cartan matrix. Here Kashiwara [9, Example 1.2.6] introduced "elementary" crystals. We follow [2, 12.3] in modifying slightly their definition which is given below.

$$B_\alpha = \{b_\alpha(m) : m \in \mathbb{N} : \forall \alpha \in \pi\}.$$

Their crystal structure is given by

$$wt\, b_\alpha(m) = -m\alpha, \varepsilon_\alpha(b_\alpha(m)) = m,$$

and

$$e_\alpha b_\alpha(m) = \left\{ \begin{array}{ll} 0 & : \; m = 0, \\ b_\alpha(m-1) & : \; m > 0, \end{array} \right.$$

$$\varepsilon_\beta(b_\alpha(m)) = -\infty, \; e_\beta(b_\alpha(m)) = f_\beta(b_\alpha(m)) = 0, \; for \; \beta \neq \alpha.$$

Since $B_\alpha$ has linear growth, we refer to it as a one dimensional crystal. Let $J$ be a countable sequence $\alpha_{i_m}, \alpha_{i_{m-1}}, \ldots, \alpha_{i_1}$, of simple roots in which each element of $\pi$ occurs infinitely many times. Then we may form the countable Cartesian product

$$\ldots \times B_{i_n} \times B_{i_{n-1}} \times \ldots \times B_{i_1},$$

where $B_{i_m}$ denotes $B_\alpha$, when $\alpha = \alpha_{im}$. We note an element b of this product simply by the sequence $\{\ldots, m_n, m_{n-1}, \ldots, m_1\}$ of its entries. Now let the $B_J$ denote the subset in which all but finitely many $m_i$ are equal to zero. Then the expression for the Kashiwara function is a finite sum and through it we obtain a crystal structure on $B_J$.

Note that $B_J$ has a distinguished element $b_\infty$, in which all the $m_i$ are equal to zero. It may also be characterized by the property that $\varepsilon_\alpha(b_\infty) = 0, \forall \alpha \in \pi$. Set $B_J(\infty) = \mathscr{F}b_\infty$.

From Kashiwara's work [8, 9] one may immediately deduce some quite remarkable facts about $B_J(\infty)$ given that $C$ is symmetrizable. These, noted in 3.7 below, can be

extended (1.6) to all classical $C$ through the Littelmann path model. We remark that if $J$ is non-redundant in the sense that every entry is non-zero for some $b \in B_J(\infty)$, then $\ldots s_{i_n} s_{i_{n-1}}, \ldots, s_{i_1}$, can be taken to be reduced decompositions of a sequence of elements of $W$.

N.B. Here there is a small but annoying subtlety (see [4, 3.13]) which one may only notice when one gets down to nitty gritty calculations. In the formulation of Kashiwara a factor of $B(\infty)$ (of which only $b_\infty$ is needed) is carried to the left. This ensures that the values of the $\varepsilon_\alpha$ stay non-negative on the elements of $B_J(\infty)$, which in turn is needed for the properties described in 3.7 to hold. Alternatively one may add "dummy" factors causing some redundancy. Thus for any $s_\alpha : \alpha \in \pi$ which occurs only finitely many times in the above sequence one adds to $B_J$ one additional factor of $B_\alpha$ at any place to their left. On this the corresponding entry stays zero for all $b \in B_J(\infty)$.

3.7.   Fix $\alpha \in \pi$. After Kashiwara a crystal B is called $\alpha$-upper normal if

$$\varepsilon_\alpha(b) = max\{n | e_\alpha^n b = 0\}, \quad \forall b \in B.$$

A crystal is called upper normal if it is $\alpha$-upper normal for all $\alpha \in \pi$. For example $B_\alpha$ is $\alpha$-upper normal; but not upper normal. Though we barely need this we remark that a crystal $B$ is said to be lower normal if for all $\alpha \in B$ one has

$$\varphi_\alpha(b) = max\{n | f_\alpha^n b = 0\}, \quad \forall b \in B,$$

where $\varphi_\alpha(b) = \varepsilon_\alpha(b) + \alpha^\vee(wtb)$. A crystal is said to be normal if it is both upper and lower normal.

Let $B$ be a subset of $B_J$ viewed as a subgraph by just retaining all edges joining elements of $B$. Because the weights of B must lie in $-\mathbb{N}\pi$, the subset $B^\mathscr{E} := \{b \in B | e_\alpha b = 0, \forall \alpha \in \pi\}$ is non-empty. Upper normality of $B$ implies that $B^\mathscr{E} = \{b_\infty\}$ by the remark in 3.6.

The first remarkable result of Kashiwara is that $B_J(\infty)$ is upper normal. By the remarks above this immediately implies that $B_J(\infty)$ is a strict subcrystal of $B_J$.

The second remarkable result of Kashiwara is that as a crystal (or equivalently as a graph) $B_J(\infty)$ is independent of $J$. We denote it by $B(\infty)$.

The third remarkable result of Kashiwara is that

$$chB(\infty) := \sum_{b \in B(\infty)} e^{wt\ b} = \prod_{\alpha \in \Delta^+} (1 - e^\alpha)^{-m_\alpha},$$

where $\Delta^+$ denotes the set of positive roots of the corresponding Kac-Moody algebra and $m_\alpha$ the multiplicity of $\alpha$ root space. We remark that in order to extend this result to the non-symmetrizable case we need the Littelmann character formula for $B(\infty)$, which expresses the latter as an alternating sum over $W$, together with the

corresponding Weyl denominator formula. The validity of the latter for the non-symnmetrizable case was established independently by Mathieu [14] and Kumar [11].

3.8. Suppose now that $C$ admits off-diagonal entries with values in $M$ as defined in 3.5. Then we modify the elementary crystals to take account of the additional elements $e_{\alpha,i}, f_{\alpha,i}$, introduced there. For this we set $\mathbf{g} = (g_1, g_2, \ldots, g_s)$ and let $\mathbf{m} = (m^1, m^2, \ldots, m^s)$ denote an arbitrary element of $\mathbb{N}^s$ setting

$$\mathbf{g}.\mathbf{m} = \sum_{i=1}^{s} g_i m^i.$$

Now define

$$B_\alpha = \{b_\alpha(\mathbf{m}) : \mathbf{m} \in \mathbb{N}^s\},$$

given the obvious crystal structure extending 3.6. In particular

$$wt\ b_\alpha(\mathbf{m}) = -(\mathbf{g}.\mathbf{m})\alpha,\ \ \varepsilon_\alpha(b_\alpha(\mathbf{m})) = (\mathbf{g}.\mathbf{m}),$$

$$e_\alpha b_\alpha(\mathbf{m}) = \begin{cases} 0 & :\ m^i = 0, \\ b_\alpha(\mathbf{m} - \mathbf{g}_i) & :\ m^i > 0. \end{cases}$$

Notice for example that the $f_{\alpha,i} : i = 1, 2, \ldots, s$, commute on $B_\alpha$. Via the insertion rules interpreted through 3.5 they also commute on $B_J$ and hence on $B_J(\infty)$. Note that $B_\alpha$ is now an s-fold tensor product of one-dimensional crystals, so we call it an s-dimensional crystal. It is not a priori obvious that this new $B_J(\infty)$ will still retain all the good properties described in 3.7.

3.9. For each $\alpha \in \pi$, define $\mathbb{Z}$ linear maps on $P$ by

$$s_{\alpha,i}\lambda = \lambda - \alpha^\vee(\lambda)_i g_i \alpha : i = 1, 2, \ldots, s,$$

where as before the $i^{th}$ subscript denotes projection onto the $i^{th}$ component of $M$, which we recall has values in $\mathbb{Z}$. One checks that

$$s_{\alpha,i}s_{\alpha,j}\lambda = \lambda - \alpha^\vee(\lambda)_j g_j \alpha - \alpha^\vee(\lambda)_i g_i \alpha + 2\delta_{i,j} g_i \alpha^\vee(\lambda)_j,$$

where $\delta$ is the Kronecker delta. Thus the $s_{\alpha,i}$ commute pairwise, are involutions and their product is $s_\alpha$. This commutation property parallels that noted in 3.8. Let $W^a$ denote the group generated by the $< s_{\alpha,i} : i = 1, 2, \ldots, s >$. We call it the augmented Weyl group $W^a$. A priori its structure depends on the choice of generators for $M$. One can ask if choices can be made so that $W^a$ is a Coxeter group and finite whenever $W$ is finite. We investigate these questions in the rank 2 case, that is when $|\pi| = 2$. Apart from the pentagonal case (see Section 5) where we discovered inadvertently that $W^a$ is the Weyl group for $\mathfrak{sl}(5)$, our reasoning is somewhat a posteriori.

## 4. Rank Two

4.1.  Fix $J$ as in 3.6 and recall the definition of $B_J(\infty)$. As noted in 3.7 it may be viewed as a presentation of the Kashiwara crystal $B(\infty)$ which is in turn a combinatorial manifestation of a Verma module (or more properly its $\mathscr{O}$ dual). As a set $B_J(\infty)$ is completely determined by its specification as a subset of countably many copies of $\mathbb{N}$. We would like to determine this subset explicitly. This seems to be rather difficult; yet Kashiwara [9, Prop. 2.2.3] found an elegant solution which we derive below by a different method which is applicable to the pentagonal crystal.

4.2.  Take $\pi = \{\alpha_1, \alpha_2\}$. Then the off-diagonal elements of $C$ are $a := -\alpha_1^\vee(\alpha_2), a' := -\alpha_2^\vee(\alpha_1)$, which are integers $\geq 0$. Set $y = aa'$. Since we may now care about preserving integrality we do not yet symmetrize $C$ as in 2.1.

There are just two possible non-redundant choices for $J = \{\ldots.j_n, j_{n-1}, \ldots, j_1\}$. The first is given by
$$j_k = \begin{cases} 1 & : \ k \ odd, \\ 2 & : \ k \ even. \end{cases}$$

The second by interchange of $1, 2$. We consider just the first.

Suppose $y = 0$. Then one easily checks that $B_J(\infty) = \mathbb{N}^2$. Assume $y > 0$. Set
$$c_k = \begin{cases} a & : \ k \ odd, \\ a' & : \ k \ even. \end{cases}$$

Define the rational functions $T_n : n \geq 3$ by $T_3(y) = 1$ and
$$T_{n+1}(y) = 1 - \frac{1}{y T_n(y)} : \forall n \geq 3.$$

Recalling the conventions of 3.6 let $\mathbf{m} = \{\ldots, m_n, m_{n-1}, \ldots, m_1\}$ denote an element of $B_J$.

**Lemma.** *One has* $\mathbf{m} \in B_J(\infty)$ *if and only if* $m_n \leq c_n T_n(y) m_{n-1}, \forall n \geq 3$.

*Proof.* It suffices to show that the subset of $B_J(\infty)$ defined by the right hand side of the lemma is $\mathscr{E}$ stable, $\mathscr{F}$ stable and that its only element annihilated by $\mathscr{E}$ is $b_\infty$.

For $n$ odd $\geq 1$ one has
$$r_{\alpha_1}^{n+2}(\mathbf{m}) - r_{\alpha_1}^n(\mathbf{m}) = a m_{n+1} - m_{n+2} - m_n.$$

Recall that if $e_{\alpha_1}(\text{resp } f_{\alpha_1})$ enters $\mathbf{m}$ at the $n^{th}$ place then the above expression must be $< 0$ (resp. $\leq 0$).

Assume the inequality of the lemma for $n$ replaced by $n + 2$, whenever $n \geq 1$. Suppose $e_{\alpha_1}$ enters at the $n^{th}$ place. If $m_n = 0$, then $e_{\alpha_1}\mathbf{m} = 0$. Otherwise $m_n$ is reduced by one. Yet
$$m_n - 1 \geq a m_{n+1} - m_{n+2} \geq a(1 - T_{n+2}(y))m_{n+1} = \frac{m_{n+1}}{a' T_{n+1}(y)}.$$

It follows that the right hand side of the lemma is satisfied by $e_{\alpha_1}\mathbf{m}$. A similar result holds for n even. This establishes stability under $\mathscr{E}$. A very similar argument establishes stability under $\mathscr{F}$.

Finally assume $e_{\alpha_1}\mathbf{m} = e_{\alpha_2}\mathbf{m} = 0$. Suppose $\mathbf{m} \neq b_\infty$ and let $m_n$ be the last non-vanishing entry of $\mathbf{m}$. One easily checks that $n \geq 3$. Then the inequalities of the lemma force $m_i > 0$, for all $i < n$. Assume n even. Then the above vanishing implies that $e_{\alpha_2}$ goes in at the $(n+2)^{nd}$ place which through the above expression for the differences of Kashiwara functions gives the contradiction $-m_n \geq 0$. The case of n odd is similar. $\square$

4.3. One may easily compute the rational functions $T_n(y)$ for small n. One obtains

$$T_3 = 1, \ T_4 = \frac{y-1}{y}, T_5 = \frac{y-2}{y-1}, \ T_6 = \frac{y^2 - 3y + 1}{y(y-2)}, \ T_7 = \frac{(y-3)(y-1)}{y^2 - 3y + 1},$$
$$T_8 = \frac{y^3 - 5y^2 + 6y - 1}{y(y-3)(y-1)}, \quad T_9 = \frac{(y-2)(y^2 - 4y + 2)}{y^3 - 5y^2 + 6y - 1}.$$
.

One may easily check the

**Lemma.** *If $y$ is real and $\geq 4$, then $1 > T_n(y) > \frac{1}{2}, \forall n > 3$.*

4.4. The $T_n(y)$ are related to the Chebyshev polynomials $P_m(x)$ defined in 2.2 by the formula
$$T_n(x^2) = \frac{P_{n-2}(x)}{xP_{n-3}(x)}, \forall n > 3. \tag{$*$}$$

4.5. In applying $(*)$ to Lemma 4.2, we recall that the case $x = 0$ has been excluded. Moreover if $P_{n-3}(x) = 0$, for a chosen value of $x$, then by Lemma 4.2 one has $m_{n-1} = 0$ and so $m_n = 0$ also. Apart from the case $x = 0$ corresponding to $\pi$ of type $A_1 \times A_1$, the remaining cases when $B(\infty)$ lies in a finite tensor product are when $P_{n-2}(x)$ has a zero for some positive integer value of $y = x^2$. By 4.3 we must have $y < 4$. Thus there are three such possible values of $y$, namely $1, 2, 3$ and these respectively are zeros of $T_4, T_5, T_7$. They correspond to types $A_2, B_2, G_2$.

4.6. One can ask if real positive non-integer zeros of $P_n(x)$ can also lead to $B_J(\infty)$ being embedded in a finite tensor product. The first interesting case is $P_4(x)$, namely the fifth Chebyshev polynomial in our conventions. One has
$$P_4(x) = x^4 - 3x^2 + 1 = (x^2 - x - 1)(x^2 + x - 1).$$

The positive roots of this polynomial are the Golden Section $g$ and its inverse. One may remark that $y = g^2 = 2.618\ldots$, and lies between 2 and 3, the latter respectively corresponding to types $B_2$ and $G_2$. From the point of view of the Weyl group (see

2.3) which is isomorphic to $\mathbb{Z}_5 \ltimes \mathbb{Z}_2$ in this case, we might expect $B_J(\infty)$ to be a strict subcrystal of a five-fold tensor product of elementary crystals. As pointed out in 3.4 some modifications are necessary since the Kashiwara functions will not be integer-valued. Now $M := \mathbb{Z}[g]$ is a free rank 2 module. Thus each elementary crystal should itself be a tensor product of one-dimensional crystals and so ultimately $B_J(\infty)$ should lie in a ten-fold tensor product of one-dimensional crystals. From this one may anticipate that the underlying root system should have ten positive roots coming in five collinear pairs of relative length $g$. To realize this we shall make two choices which ultimately may affect the result, namely we choose $g_1 = 1, g_2 = g$ in 3.5 and the Cartan matrix to have off-diagonal entries both equal to $-g$. Remarkably the resulting root system is stable under the augmented Weyl group (3.9) itself isomorphic to $S_5$, the permutation group on five symbols.

## 5. The Pentagonal Crystal

5.1.  Recall the notation and conventions of 2.2 and take the Cartan matrix to have off-diagonal entries equal to $-g$, where $g$ is the Golden Section. Choose $J$ as in 4.2 with $B_\alpha$, $B_\beta$ the elementary crystals defined in 3.8 with $g_1 = 1, g_2 = g$. We write $e_{\alpha,1} = e_\alpha$, $e_{\alpha,2} = e_{g\alpha}$, and so on. The entries in the $i^{th}$ factor of $B_J$ will be denoted by $\mathbf{m}_j = (m_j, n_j)$.

5.2.  The aim of this section is to give an explicit description of $B_J(\infty)$ as a subset of $B_J$ in a manner analogous to 4.2. In particular we show that $B_J(\infty)$ lies in a five-fold tensor product of elementary crystals. We show that as a crystal it is independent of the two possible non-redundant choices of $J$. We denote the resulting crystal by $B(\infty)$. We show that the formal character of $B(\infty)$ has ten factors each corresponding to the ten positive roots alluded to in 4.6.

5.3.  We may regard $B_J$ as a repeated tensor product of the crystal $B_\beta \otimes B_{g\beta} \otimes B_\alpha \otimes B_{g\alpha}$ defined via 3.6, 3.7. Set $\mathbf{m} = (m, n)$. A given element $b \in B_J$ is given by a finite sequence $(\mathbf{m}_j, \ldots, \mathbf{m}_1) := (m_j, n_j, m_{j-1}, n_{j-1}, \ldots, m_1, n_1)$ of non-negative integers, though $j$ may be arbitrarily large.

Recall 3.5 that the Kashiwara functions take values in $\mathbb{Z} \oplus \mathbb{Z}g$. Their components in the first (resp. second) factor will be denoted by $r_\alpha^j, r_\beta^j$ (resp. $r_{g\alpha}^j, r_{g\beta}^j$). These integers determine the places in $b$ at which the crystal operators enter via the rules $i), ii)$ of 3.3. As in 4.2, it is only certain differences that matter.

Take $b = (\mathbf{m}_j, \ldots, \mathbf{m}_1)$ defined as above. For $j$ odd $\geq 1$, we obtain $r_\alpha^{j+2}(b) - r_\alpha^j(b)$ (resp. $r_{g\alpha}^{j+2}(b) - r_{g\alpha}^j(b)$) as the coefficient of 1 (resp. $g$) in the expression

$$g(\mathbf{m}_{j+1}.\mathbf{g}) - (\mathbf{m}_{j+2}.\mathbf{g}) - (\mathbf{m}_j.\mathbf{g}) = g(m_{j+1} + gn_{j+1}) - (m_{j+2} + gn_{j+2}) - (m_j + gn_j).$$

Then the identity $g^2 = g + 1$ gives

$$r_\alpha^{j+1}(b) - r_\alpha^j(b) = n_{j+1} - m_{j+2} - m_j, \; r_{g\alpha}^{j+2}(b) - r_{g\alpha}^j(b) = m_{j+1} + n_{j+1} - n_{j+2} - n_j.$$

When $j$ is even $\geq 2$, the above formulae still hold but with $\alpha$ replaced by $\beta$.

**5.4.** At first one might expect the exact analogue of 4.2 to hold with the inequality

$$(\mathbf{m}_j.\mathbf{g}) \leq gT_j(g^2)(\mathbf{m}_{j-1}.\mathbf{g}), \qquad\qquad (*)$$

suitably interpreted. Since $T_6(g^2) = 0$ this would give $B_J(\infty)$ to lie in a five-fold tensor product of two dimensional crystals. The true solution is more complex. It is given by the following

**Proposition.** *One has* $(\mathbf{m}_j, \ldots, \mathbf{m}_1) \in B_J(\infty)$ *if and only if*
  $(i)$ $m_k, n_k = 0 : k \geq 6$.
  $(ii)$ $m_3 = n_2 - u, n_3 = m_2 + n_2 - v : u, v \geq 0$.
  $(iii)$ $m_4 = n_2 - v - s, n_4 = m_2 + n_2 - u - v - t :$
      $u + t \geq 0, v + s \geq 0, v + t \geq 0, s + t + v \geq 0$.
  $(iv)$ $m_5 = m_2 - v - t - a, n_5 = n_2 - u - v - s - t - a' :$
      $v + t + a \geq 0, u + t + a' \geq 0, s + v + t + a \geq 0,$
      $s + v + t + a' \geq 0, s + t + v + a + a' \geq 0$.

Remark 1 It is implicit that $m_k, n_k \geq 0$, for all $k \in \mathbb{N}^+$, and this gives some additional inequalities.

Remark 2. Recall that $T_3(g^2) = 1$. Interpret $m + gn \leq m' + gn'$ to mean $m \leq m', n \leq n'$. Then $(ii)$ can be interpreted as $(*)$ for k = 3. However $(iii)$ and $(iv)$ cannot be similarly interpreted.

Remark 3. Set $b = (\mathbf{m}, \mathbf{n})$. Notice that

$$r_\beta^6(b) - r_\beta^4(b) = n_5 - m_4 = -(u + t + a') \leq 0,$$

$$r_{g\beta}^6(b) - r_{g\beta}^4(b) = m_5 + n_5 - n_4 = -(v + s + t + a + a') \leq 0.$$

These imply $(i)$ above.

Remark 4. In 5.9 we describe an algorithm giving these inequalities.

**5.5.** The proof of Proposition 5.4 follows exactly the same path as the proof of Lemma 4.2, as do also the remaining assertions in 5.2. However verification of the details should only be attempted by a certified masochist. We give a few vignettes from the proof. These illustrate the ubiquitous nature of the inequalities.

5.6.  Define $B_J(\infty)$ by the inequalities in the right hand side of Proposition 5.4. We shall first illustrate stability under $\mathscr{E}$ in the more tricky cases.

Take $b = (\mathbf{m}_j, \dots, \mathbf{m}_1)$ as before and suppose that $e_\alpha$ enters $b$ at the third place. This implies in particular that

$$0 > r_\alpha^5(b) - r_\alpha^3(b) = n_4 - m_5 - m_3 = a.$$

Moreover if $e_\alpha b \geq 0$, then $m_3 \geq 1$ and this insertion decreases $m_3$ by 1. The latter can be achieved without changing the remaining entries in $b = (\mathbf{m}_j, \dots, \mathbf{m}_1)$ by increasing $u$ and $a$ by 1 and decreasing $t$ by 1. Then the only inequalities for the new variables that might fail are those involving $t$ but neither $a$ nor $u$. One is thus reduced to the expressions $v + t, v + t + s + a', s + t + v$. Yet in the old variables all these expressions are $> 0$ in view of the fact that $a + v + t, a + v + t + s + a', a + s + t + v$ are all non-negative, whilst as we have seen above $a < 0$.

Suppose as a second example that $e_\beta$ enters $b$ in the fourth place. This implies in particular that

$$0 > r_\beta^6(b) - r_\beta^4(b) = n_5 - m_6 - m_4 = -(u + t + a').$$

Moreover if $e_\beta b \neq 0$, then $m_4 \geq 1$ and the insertion of $e_\beta$ decreases $m_4$ by 1. The latter can be achieved without changing the remaining entries in $b = (\mathbf{m}_j, \dots, \mathbf{m}_1)$ by increasing $s$ by 1 and decreasing $a'$ by 1. Then the only inequalities for the new variables that might fail are those involving $a'$ but not $s$. This is just $u + t + a'$ which is strictly positive by the above.

5.7.  Retain the above conventions. We illustrate stability under $\mathscr{F}$ in some of the more tricky situations.

Suppose that $f_\alpha$ enters $b$ at the third place. This implies in particular that

$$0 < r_\alpha^3(b) - r_\alpha^1(b) = n_2 - m_3 - m_1 = u - m_1,$$

and so $u > m_1 \geq 0$.

Moreover this insertion increases $m_3$ by 1. The latter can be achieved without changing the remaining entries in $b = (\mathbf{m}_j, \dots, \mathbf{m}_1)$ by decreasing $u$ and $a$ by 1 and increasing $t$ by 1. One easily checks that all inequalities of Proposition 5.4 are preserved.

As a second example suppose that $f_{g\beta}$ enters $b$ at the fourth place. This implies in particular that

$$0 < r_{g\beta}^4(b) - r_{g\beta}^2(b) = m_3 + n_3 - n_4 - n_2 = t.$$

Moreover this insertion increases $n_4$ by 1. The latter can be achieved without changing the remaining entries in $b = (\mathbf{m}_j, \dots, \mathbf{m}_1)$ by decreasing $t$ by 1 and increasing $a$ and $a'$ by 1. Since already $u, v \geq 0$ and $t > 0$, it easily follows that all inequalities are preserved.

5.8. Retain the conventions of 5.6. To complete the proof of Proposition 5.4 it remains to show that $B_J(\infty)^{\mathscr{E}} = \{b_\infty\}$. This obtains from the following lemma which also implies that $B_J(\infty)$ is upper normal.

**Lemma.** *For all $b \in B_J(\infty)$, one has*
   *(i) $e_\alpha b = 0$, if and only if $a \geq 0, a + u - m_1 \geq 0, m_5 = 0$,*
   *(ii) $e_{g\alpha} b = 0$, if and only if $a' \geq 0, a' + v - n_1 \geq 0, n_5 = 0$,*
   *(iii) $e_\beta b = 0$, if and only if $s \geq 0, u + t + a' = 0$,*
   *(iv) $e_{g\beta} b = 0$, if and only if $t \geq 0, a + a' + s + t + v = 0$.*

*Proof.* Suppose $e_\alpha b = 0$. This means that there exists $j \in \mathbb{N}$ such that $e_\alpha$ enters $b$ at the $(2j + 1)^{th}$ place and that $m_{2j+1} = 0$. One has

$$(r_\alpha^5, r_\alpha^3, r_\alpha^1) = (m_5, m_5 - a, m_5 - a + m_1 - u).$$

Now $u \geq 0$, so then $e_\alpha$ cannot enter at the first place since this would mean that $m_1 = 0$. Suppose $a < 0$. Then $e_\alpha$ enters $b$ at the third place implying that $0 = m_3 = n_2 - u = 0$. Since $n_5 \geq 0$, we obtain $v + s + t + a' \leq 0$, and since $a < 0$, this contradicts $(iv)$ of 5.4. Thus $a \geq 0$ and so $e_\alpha$ enters at the fifth place. All this gives $(i)$. The proof of $(ii)$ is practically the same.

Suppose $e_\beta b = 0$. This means that there exists $j \in \mathbb{N}^+$ such that $e_\beta$ enters $b$ at the $2j^{th}$ place and that $m_{2j} = 0$. One has

$$(r_\beta^6, r_\beta^4, r_\beta^2) = (0, u + t + a', u + t + a' - s).$$

Recall that $u + t + a' \geq 0$. Suppose $s < 0$. Then $e_\beta$ enters $b$ at the second place implying $m_2 = 0$. Since $m_5 \geq 0$, we obtain $v + t + a = 0$, and since $s < 0$, this contradicts $(iv)$ of 5.4. Thus $s > 0$. If $u + t + a' > 0$, then $0 = m_4 = n_2 - v - s$. Yet $n_5 \geq 0$, and this forces a contradiction. All this gives $(iii)$. The proof of $(iv)$ is practically the same. □

5.9. Let $J'$ be the second non-redundant sequence described in 4.2. We sketch briefly how to obtain a crystal isomorphism $\varphi : B_J(\infty) \xrightarrow{\sim} B_{J'}(\infty)$.

Let us use $F_\alpha^{\mathbf{m}}$ to denote $f_\alpha^m f_{g\alpha}^n$, when $\mathbf{m} = (m, n)$ and $E_\alpha b = 0$ to mean $e_\alpha b = e_{g\alpha} b = 0$. Similar meanings are given to these expressions when $\beta$ replaces $\alpha$.

Then $f := F_\alpha^{\mathbf{m}_1} F_\beta^{\mathbf{m}_2} F_\alpha^{\mathbf{m}_3} F_\beta^{\mathbf{m}_4} F_\alpha^{\mathbf{m}_5}$ is said to be in normal form if

$$E_\alpha F_\beta^{\mathbf{m}_2} F_\alpha^{\mathbf{m}_3} F_\beta^{\mathbf{m}_4} F_\alpha^{\mathbf{m}_5} b_\infty = 0, E_\beta F_\alpha^{\mathbf{m}_3} F_\beta^{\mathbf{m}_4} F_\alpha^{\mathbf{m}_5} b_\infty = 0, \ldots, E_\alpha b_\infty = 0.$$

Let $\mathscr{F}_0 \subset \mathscr{F}$ denote the subset of elements having normal form. Obviously for each element of $B_J(\infty)$ can be written as $f b_\infty$, for some unique $f \in \mathscr{F}_0$. (Nevertheless it should be stressed that the definition of normal form depends on a choice of reduced decomposition.)

Take $f$ as above with $\mathbf{m}_j = (m_j, n_j) : j \in \mathbb{N}^+$, given by the expressions in $(i) - (iv)$ of 5.4. One checks that the condition for $f$ to have normal form exactly reproduces the inequalities of Proposition 5.4. Moreover $fb_\infty$ takes exactly the form of the element $b$ occurring in the proposition, namely $(\mathbf{m}_5, \ldots, \mathbf{m}_1)$. In the classical Cartan case this is essentially Kashiwara's algorithm (cf [9, see after Cor. 2.2.2] for computing $B_J(\infty)$; but it is not too effective as each step becomes increasingly arduous. Again in our more general set-up it was not a priori obvious that such a procedure would work. The explanation of its success obtains from Section 7.

5.10.  Now let $b'_\infty$ denote the canonical generator of $B_{J'}(\infty)$ of weight zero. We define the required crystal map $\varphi$ of 5.9 by setting $\varphi(fb_\infty) = fb'_\infty$, for all $f \in \mathscr{F}_0$. One checks that $f$ still has normal form with respect to $b'_\infty$. (This is a slightly different calculation to that given in 5.9; but still uses exactly the same inequalities of Proposition 5.4.) It follows that $\varphi$ is injective. Repeating this procedure with $J$ and $J'$ interchanged we obtain crystal embeddings $B_J(\infty) \hookrightarrow B_{J'}(\infty) \hookrightarrow B_J(\infty)$. Since these preserve weight and weight subsets have finite cardinality, they must both be isomorphisms, as required.

5.11.  Recall 3.5. Then just as in [3], 5.3.13, we may define a singleton highest weight crystal $S_\lambda = \{s_\lambda\}$ of weight $\lambda$ with $\varepsilon_\alpha(s_\lambda) = -\alpha^\vee(\lambda)$, for all $\lambda \in P^+$ and one checks that $B(\lambda) = \mathscr{F}(b \otimes s_\lambda)$ is a strict subcrystal of $B(\infty) \otimes S_\lambda$. The upper normality of $B(\infty)$ implies that $B(\lambda)$ is a normal crystal (see [2], 5.2.1, for example) and moreover its character can be calculated in a manner analogous to the case of a classical Cartan matrix (see [2],6.3.5, for example) using appropriate Demazure operators. This leads to the character formula for $B(\infty)$ alluded to in 5.2. However one may now begin to suspect that $B(\infty)$ is itself just a Kashiwara crystal for a classical Cartan matrix of larger rank. This is what we shall show in Sections 6 and 7.

## 6. The extended Weyl group for the Pentagonal Crystal

6.1.  Take $\pi$ and $C$ as in 5.1. One checks that W applied to $\pi$ generates as a single orbit the set $\Delta_s$, called the set of short roots, given by the formulae below

$$\Delta_s = \Delta_s^+ \sqcup \Delta_s^-, \Delta_s^- = -\Delta_s^+, \Delta_s^+ = \{\alpha, \alpha + g\beta, g\alpha + g\beta, g\alpha + \beta, \beta\}.$$

Set $\Delta_\ell = g\Delta_s$, called the set of long roots. They are collinear to the short roots. Finally set $\Delta = \Delta_s = \Delta_s \sqcup \Delta_\ell$.

6.2.  At first sight it may seem that the introduction of $\Delta_\ell$ is quite superfluous, yet it is rather natural from the point of view of the theory of crystals. Indeed we have already noted that we need both short and long simple roots to interpret the

Kashiwara tensor product and then to obtain $B(\infty)$. Again the analysis of 5.11 implies that

$$chB(\infty) = \prod_{\gamma \in \Delta^+} (1 - e^{-\gamma})^{-1}$$

As promised we shall eventually prove the above formula by slightly different means. However for the moment we note that it may in principle be obtained from the inequalities in 5.4. Here we have ten parameters whose values define a subset of $\mathbb{N}^{10}$. It is possible to break this subset into smaller "sectors" in which each of the corresponding terms become a geometric progression. In general this is a rather impractical procedure and indeed even in the present case the number of sectors runs into several thousand. However one can at least deduce that $chB(\infty)$ must be a rational function of the variables $e^\alpha, e^{g\alpha}, e^\beta, e^{g\beta}$. Moreover using 5.8, one may deduce that the subset $B^\alpha$ of $B_J(\infty)$ defined by the condition $\mathbf{m}_1 = 0$ has an $s_\alpha$ invariant character. Yet $B_J(\infty) \cong B^\alpha \otimes B_\alpha$, and so $chB_J(\infty)$ is invariant with respect to an obvious translated action of $s_\alpha$. Through the isomorphism $B_J(\infty) \xrightarrow{\sim} B_{J'}(\infty)$, described in 5.9, 5.10, it follows that the common crystal admits a formal character which is invariant under the translated action of $W$. A slightly more refined analysis shows it to be invariant under a translated action of the augmented Weyl group $W^a$.

In the present "finite" situation it is more efficient to use the Demazure operators as alluded to 5.11. However in general there is no known combinatorial recipe to obtain an expression for $chB(\infty)$ as an (infinite) product. This is mainly because there is no known meaning to imaginary roots, outside their Lie algebraic definition. In the non-symmetrizable Borcherds case such a product formula is not even known, even though $B(\infty)$ can be defined and its formal character calculated [5, Thm. 9.1.3].

6.3.  Recall 3.9. It is rather easy to compute the image of a generator of $W^a$ on a given root, always remembering however that its elements are only $\mathbb{Z}$-linear maps. We need only do this for the $s_{\alpha,i} : i = 1, 2$, since the action of the remaining elements can be obtained by $\alpha, \beta$ interchange. The result is given by the following

**Lemma.**
   (i) $s_{\alpha,1}$ *stabilizes* $g\alpha, \beta, g\alpha + \beta$, *and interchanges elements in each of the pairs* $(\alpha, -\alpha); (\alpha + g\beta, g\beta); (g\alpha + g\beta, g^2\alpha + g\beta); (g\alpha + g^2\beta, g^2(\alpha + \beta))$.
   (ii) $s_{\alpha,2}$ *stabilizes* $\alpha, g^2(\alpha + \beta), g\alpha + g^2\beta$, *and interchanges elements in each of the pairs* $(g\alpha, -g\alpha); (\alpha + g\beta, g^2\alpha + g\beta); (g\alpha + \beta, \beta); (g\beta, g(\alpha + \beta))$.

6.4.  We see that $s_{\alpha,1}$ can interchange long and short roots. Since is $\Delta$ just two $W$ orbits it follows that $\Delta$ is a single $W^a$ orbit. The stability of $\Delta$ under $W^a$ came to us as a surprise. It truth depends on the particular basis we have chosen for $M$.

6.5.   A further surprise is the following. Let $\pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the simple roots of a system of type $A_4$ in the standard Bourbaki [1] labelling. Make the identifications $\alpha_1 = \alpha, \alpha_2 = g\beta, \alpha_3 = g\alpha, \alpha_4 = \beta$. Then the relations in 6.3 allow us to make the further identifications $s_{\alpha_1} = s_{\alpha,1}, s_{\alpha_2} = s_{\beta,2}, s_{\alpha_3} = s_{\alpha,2}, s_{\alpha_4} = s_{\beta,1}$. We conclude that $W^a$ is actually the Weyl group $W(A_4)$ of the system of type $A_4$ with the augmented set of simple roots $(\alpha, g\beta, g\alpha, \beta)$ being the set of simple roots of this larger rank system. However notice that $s_\alpha$ (resp. $s_\beta$) is not a reflection in this larger system; but rather a product of commuting reflections, namely: $s_{\alpha_1} s_{\alpha_3}$ (resp. $s_{\alpha_2} s_{\alpha_4}$). Precisel;y what we obtain is the following

**Lemma.**
    (i)   The map $\{s_{\alpha,1}, s_{\beta,2}, s_{\alpha,2}, s_{\beta,1}\} \rightarrowtail \{s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}\}$ extends to an isomorphism $\psi$ of $W^a$ onto $W(A_4)$ of Coxeter groups. In particular

$$\psi(s_\alpha) = s_{\alpha_1} s_{\alpha_3}, \quad \psi(s_\alpha) = s_{\alpha_2} s_{\alpha_4}.$$

    (ii)   Set $\widehat{W} = W^a \xrightarrow{\sim} W(A_4)$. The map $\{\alpha, g\beta, g\alpha, \beta\} \rightarrowtail \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ extends to an isomorphism of $\mathbb{Z}\widehat{W}$ modules.

**Remark**. Thus the root diagram of $A_4$ can be drawn on the plane. This is the beautiful picture presented in Figure 1, which illustrates a tiling using two triangles based on the Golden Section. It not only admits pentagonal symmetry coming from $W$; but also a further "hidden" symmetry coming from $\widehat{W}$. Similar considerations apply to weight diagrams for $A_4$. In particular the defining (five-dimensional) module for $A_4$ gives rise to what we call a zig-zag triangularization of the regular pentagon, see Figure 2. The fact that the longer to shorter length ratio of the triangles obtained is the Golden Section now follows from the above lemma! Moreover the angles in the triangles are thereby easily computed. In Section 8 we describe the generalization of the above lemma to type $A_{2n}$ and its consequences for aperiodic tiling based on the $n$ triangles obtained from appropriate triangularizations of the regular $(2n + 1)$-gon (see Figure 2).

6.6.   There is a relation of the Golden Section to the dodecahedron which goes back to ancient times. As a consequence it is not too surprising that our root system can be obtained from the vertices of the dodecahedron by projection onto the plane of one of the faces (see Figure 1). This is an orthogonal projection as the remaining co-ordinate is normal to the face in question. Let us describe the co-ordinates of the vertices of dodecahedron $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ which thus project onto $\{\alpha, g\beta, g\alpha, \beta\}$.
    The planar co-ordinates of $\alpha$ and $\beta$ which generate our pentagonal system can be taken to be

$$\alpha = (1, 0), \quad \beta = (-g/2, \sqrt{1 - g^2/4}).$$

View these as two vertices, called $\alpha_1$ and $\alpha_4$, of one face $f_1$ of the dodecahedron. Now project the dodecahedron onto $f_1$. Of course this takes all the vertices of the dodecahedron onto the plane defined by $f_1$. Exactly one of these becomes collinear to $\alpha$ (resp. $\beta$) and this with relative length of exactly g. Call it $\alpha_3$ (resp. $\alpha_2$). Now the perpendicular distance of $f_1$ to its opposite face $f_2$ is exactly $(g+1)$. We take this normal direction to these two faces as defining a third co-ordinate fixed to be zero at the centre of the dodecahedron. Thus the value of this co-ordinate on $f_1$ equals $(g+1)/2$, whilst its value on $\alpha_3$ and on $\alpha_2$ turns out to be $(g-1)/2$. We conclude that the co-ordinates of these four vertices of the dodecahedron are given by

$$\alpha_1 = (1, 0, (g+1)/2), \quad \alpha_2 = (-g^2/2, g\sqrt{1 - g^2/4}, (g-1)/2),$$
$$\alpha_3 = (g, 0, (g-1)/2), \quad \alpha_4 = (-g/2, \sqrt{1 - g^2/4}, (g+1)/2).$$

These co-ordinates allow one to calculate the scalar products between the above vertices. One finds that $(\alpha_i, \alpha_i) = 3(g+2)/4$, for all $i = 1, 2, 3, 4$, whilst $(\alpha_1, \alpha_2) = (\alpha_3, \alpha_4) = -1/2 - g/4 = -(\alpha_1, \alpha_4)$ and $(\alpha_1, \alpha_3) = (\alpha_2, \alpha_4) = 5g/4 = -(\alpha_2, \alpha_3)$. From this one obtains a further fact - there is no orthogonal projection of the set $\pi$ of simple roots of $A_4$ onto these vertices of the dodecahedron. Indeed the above vectors have all the same lengths as do also their pre-images $\hat{\alpha}_i : i = 1, 2, 3, 4$. Thus for some $a \in \mathbb{R}$ the fourth co-ordinate must be $a$ or $-a$ in each element of $\pi$. Since $(\hat{\alpha}_1, \hat{\alpha}_3) = 0$, this already forces $a^2 = 5g/4$ and then that $(\hat{\alpha}_2, \hat{\alpha}_3) = -5g/2$ and $(\hat{\alpha}_1, \hat{\alpha}_1 = 2g + 3/2$, giving the contradiction $g = 1/2$.

## 7. Beyond the Pentagonal Crystal

7.1.   The aim of this section is to show that the pentagonal crystal described above and coming from the Golden Section is in fact a manifestation of the "ordinary" Kashiwara crystal in type $A_4$ with respect to special reduced decompositions of the longest element of the Weyl group of the latter. The construction we give can be immediately generalized to the largest zeros of the $(2n+1)^{th}$ Chebyshev polynomial, though as we shall see there is a slight difference when $2n+1$ is not prime. We start with three easy (read, well-known) facts.

7.2.   For our first easy fact, let $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{2n}\}$ be the set of simple roots for a system of type $A_{2n}$ in the usual Bourbaki labelling [1, Appendix]. Set $s_i = s_{\alpha_i} : i = 1, 2, \ldots, 2n$. These are Coxeter generators of the Weyl group for type $A_{2n}$, and which we view as elementary permutations. Set $s_\alpha = s_1 s_3 \ldots s_{2n-1}, s_\beta = s_{2n} s_{2n-2} \ldots s_2$, which are involutions. Take $i \in \{1, 2, \ldots, n\}$. One checks that $s_\alpha s_\beta(2i+1) = s_\alpha(2i) = 2i-1$, whilst $s_\alpha s_\beta(2i) = s_\alpha(2i+1) = 2i+2$. Thus $s_\alpha s_\beta$ is a (2n+1)-cycle. We conclude that $< s_\alpha, s_\beta >$ is the Coxeter group isomorphic to $\mathbb{Z}_n \ltimes \mathbb{Z}_2$ as an abstract group. Its unique longest element $w_0$ has two reduced decompositions namely $(s_\alpha s_\beta)^n s_\alpha$ and

$s_\beta(s_\alpha s_\beta)^n$. Observe further that $s_\beta(s_\alpha s_\beta)^n(2i) = s_\beta(2(n+1-i)+1) = 2(n+1) - 2i$, whilst $s_\beta(s_\alpha s_\beta)^n(2i + 1) = s_\beta(2(n-i)) = 2n + 2 - (2i + 1)$. Thus $w_0$ is also the unique longest element of the Weyl group in type $A_{2n}$. Moreover substituting the above expressions for $s_\alpha, s_\beta$, it follows that $w_0$ has length $\leq (2n + 1)n$ and so the resulting expressions for $w_0$ as products of elementary permutations are again reduced decompositions in the larger Coxeter group.

7.3.   Our second easy fact concerns the factorization of the Chebyshev polynomials $P_{2n}(x) : n \in \mathbb{N}^+$. Define the polynomials

$$Q_n(x) := P_n(x) - P_{n-1}(x) : n \in \mathbb{N}. \tag{$*$}$$

These satisfy the same recurrence relations as the $P_n(x)$; but with different initial conditions which are now $Q_{-1} = 1, Q_0 = 1$. A few examples are

$$Q_1 = x - 1, \ Q_2 = x^2 - x - 1, \ Q_3 = x^3 - x^2 - 2x + 1, \ Q_4 = x^4 - x^3 - 3x^2 + 2x + 1.$$

Of these just $Q_4$ factorizes over $\mathbb{Q}$ giving $Q_4 = (x - 1)(x^3 - 3x - 1)$. It is the first case when $2n + 1$ is not prime.

**Lemma.** *For all $n \in \mathbb{N}$ one has*
   $(i)_n$   $Q_n(x)Q_n(-x) = (-1)^n P_{2n}(x),$
   $(ii)_n$   $Q_n(x)Q_{n-1}(-x) = (-1)^{n-1} P_{2n-1}(x) + (-1)^n,$
   $(ii)_n$   $Q_{n+1}(x)Q_{n-1}(-x) = (-1)^{n+1} P_{2n}(x) + (-1)^n x.$

*Proof.* The cases for which $n = 0$ are easily checked. Then one checks using the recurrence relations for the $P_n$ and $Q_n$ that $(i)_{n-1}$ and $(ii)_{n-1}$ imply $(ii)_n$, that $(i)_{n-2}$ and $(ii)_{n-1}$ imply $(iii)_{n-1}$ and finally that $(ii)_n$ and $(iii)_{n-1}$ imply $(i)_n$.     □

7.4.   Our third easy fact describes when $Q_n(x)$ is irreducible over $\mathbb{Q}$.

**Lemma.** *For all $n \in \mathbb{N}^+$ one has*
   *(i) The roots of $Q_n(x)$ form the set $\{2\cos(2t - 1)\pi/(2n + 1) : t \in \{1, 2, \ldots, n\}\}$,*
   *(ii) If $2m + 1$ divides $2n + 1$, then $Q_m(x)$ divides $Q_n(x)$,*
   *(iii) If $2n + 1$ is prime, then $Q_n(x)$ is irreducible over $\mathbb{Q}$.*

*Proof.* $(i)$. By $(*)$ of 2.2 and $(*)$ of 7.3, one has

$$(\sin\theta)Q_n(2\cos\theta) = \sin(n + 1)\theta - \sin n\theta.$$

Yet the right hand side vanishes for $\theta = (2t - 1)\pi/(2n + 1) : t \in \{1, 2, \ldots, n\}$, since $\sin(2n + 1)\theta = 0$ and $\cos n\theta = -\cos(n + 1)\theta \neq 0$ . Hence $(i)$. Then $(ii)$ follows from $(i)$ by comparison of roots.

   $(iii)$. Set $z = e^{i\theta}$, with $\theta = \pi/(2n+1)$. By (i) the roots of $Q_n(2x)$ are the real parts of $z^{2t-1} : t \in \{1, 2, \ldots, n\}$. Since $2n + 1$ is assumed prime, these are all $2(2n + 1)^{th}$ primitive roots of unity. There are therefore permuted by the Galois group of $\mathbb{Q}[z]$ over $\mathbb{Q}$ and so are their real parts. Hence they cannot satisfy over $\mathbb{Q}$ a polynomial

equation of degree $< n$. Thus $Q_n(2x)$ is irreducible over $\mathbb{Q}$ and for the same reason so is $Q_n(x)$. $\qquad\square$

7.5. Return to the notation and hypotheses of 7.2. Let $J$ be the sequence of simple roots defined by the reduced decomposition $(s_\alpha s_\beta)^n s_\alpha$ of $w_0$, namely $J = \{\alpha, \beta, \ldots, \alpha\}$. Set $g = 2\cos(\pi/(2n+1))$ which is the Golden Section if $n = 2$. Let $B_j$ denote the elementary crystal corresponding to $j^{th}$ entry in $J$ counting from the right. Thus $B_j$ is of type $\alpha$ (resp. $\beta$) if j is odd (resp. even) and let $\mathbf{m}_j$ denote its entry. Set $\mathbf{m} = \{\mathbf{m}_{2n+1}, \ldots, \mathbf{m}_1\}$, which we view as an element of $B_J$. Use the convention that $\mathbf{m}_j = 0, \forall j \notin \{1, 2, \ldots, 2n+1\}$. Then as in 4.2 successive differences of the Kashiwara function take the form

$$r_\alpha^{2j+1}(\mathbf{m}) - r_\alpha^{2j-1}(\mathbf{m}) = -\mathbf{m}_{2j+1} - \mathbf{m}_{2j-1} + g\mathbf{m}_{2j}. \qquad (*)$$

When $\alpha$ is replaced by $\beta$ then the same relation holds except that $2j+1$ is replaced by $2j$.

7.6. Retain the notation of 7.5 but now interpret $\alpha$ (resp. $\beta$) in $J$ as the sequence $\{\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}\}$ (resp. $\{\alpha_{2n}, \alpha_{2n-2}, \ldots, \alpha_2\}$), so now $B_J$ denotes the crystal which is a $2n(2n+1)$-fold tensor product of elementary crystals in type $A_{2n}$. Let $B_{i,j} : j \in \{1, 2, \ldots, 2n+1\}$ denote the elementary crystal $B_{\alpha_i} : i \in \{1, 2, \ldots, 2n\}$ occurring in the $j^{th}$ place of $J$ counting from the right, let $m_{i,j}$ denote its entry and $\mathbf{m}$ the element of $B_J$ they define. Then the successive differences of Kashiwara functions take the form

$$r_{2i+1}^{2j+1}(\mathbf{m}) - r_{2i+1}^{2j-1}(\mathbf{m}) = m_{2i+1,2j+1} - m_{2i+1,2j-1} + m_{2i,2j} + m_{2i+2,2j}, \qquad (*)$$

with $2i$ replacing $2i+1$ when $2j$ replaces $2j+1$.

7.7. We attempt to interpret $(*)$ of 7.5 so that it becomes $(*)$ of 7.6. Since $Q_n(g) = 0$ by $(i)$ of 7.4 and $Q_n$ is a degree $n$ monic polynomial with integer coefficients, it follows that the ring $\mathbb{Z}[g]$ is a free $\mathbb{Z}$ module of rank n. Let $\{g_i\}_{i=0}^{n-1}$ be a free basis for $\mathbb{Z}[g]$ with $g_0 = 1$ and using the convention that $g_{-1} = g_n = 0$. Notice that we have shifted the labelling by 1 relative to our convention in 3.8 and that we are using the same basis for all $\alpha \in \pi$. Following 3.8 we write

$$r_\alpha^{2j+1}(\mathbf{m}) = \sum_{i=0}^{n-1} g_i r_{2i+1}^{2j+1}(\mathbf{m}), \quad \mathbf{m}_{2j+1} = \sum_{i=0}^{n-1} g_i m_{2i+1,2j+1}, \quad \mathbf{m}_{2j} = \sum_{i=1}^{n} g_{n-i} m_{2i,2j+1}.$$

After these substitutions and equating coefficients of $g_i$ in $(*)$ of 7.5 we obtain $(*)$ of 7.6 given that $gg_{n-i} = g_i + g_{i-1}$, equivalently that $gg_i = g_{n-i} + g_{n-i-1}$, for all $i = 0, 1, \ldots, n-1$. Now set $p_{2i} = g_i, p_{2i+1} = g_{n-i-1}$, for all $i = 0, 1, \ldots, [n/2]$. Note that this implies $p_{-1} = 0, p_0 = 1$ and $p_n = p_{n-1}$. (We also obtain $p_{n+1} = g_0$ for $n$ even but this we ignore.) These identifications give $p_{i+1} = gp_i - p_{i-1}$ for all

$i = 0, 1, \ldots, n-1$. It follows that $p_i = P_i(g)$. In other words $p_i$ is the $i+1^{th}$ Chebyshev polynomial evaluated at g. Since the latter are monic polynomials of degee $i$, the $\{p_i : i = 0, 1, \ldots, n-1\}$ form a free basis of $\mathbb{Z}[g]$. In addition the relation $p_n = p_{n-1}$ becomes exactly the relation $Q_n(g) = 0$, precisely as required. Thus our goal has been achieved and we have shown the

**Theorem.** *Let $C$ be the $2 \times 2$ Cartan matrix with $-2\cos(\pi/2n + 1) : n \in \mathbb{N}^+$ as off-diagonal elements. Then the crystal $B(\infty)$ defined through 3.8 using the free basis $\{P_i(g)\}_{i=0}^{n-1}$ of $\mathbb{Z}[g]$ is isomorphic to the crystal $B(\infty)$ in type $A_{2n}$.*

7.8.   From 7.7 we obtain all the good properties of the pentagonal crystal noted in Section 5, namely that it is upper normal, independent of the choice of $J$ and satisfies the character formula given 6.2. We do *not* obtain the explicit description of $B_J(\infty)$ described in 5.4; but we do obtain the justification of this description given by the Kashiwara algorithm noted in 5.9.

## 8. Weight Diagrams

8.1.   Recall that Lemma 6.5 allows one to draw the weight diagrams of $A_4$ in the plane and that furthermore from the defining representation one naturally recovers the two triangles used in (aperiodic) Penrose tiling. This leads to the following question. Suppose we are given a Penrose tiling of part $P$ of the plane. When do the vertices of $P$ viewed as a graph form a weight diagram for $A_4$ ?

We note below that one may similarly draw the weight diagrams of $A_{2n}$ in the plane and then one can further ask if it is possible recover a family $\mathscr{T}_{2n+1}$ of triangles (see 8.4) which lead to higher aperiodic tiling. Then one can similarly ask which such tilings are weight diagrams.

Conversely given a weight diagram viewed as a set of points on the plane can one join vertices to obtain a tiling in (part of) the plane using just the elements of $\mathscr{T}_{2n+1}$ or possibly a slightly bigger set ? In fact the image of the weight lattice of $A_{2n}$ for $n \geq 2$ is dense in the plane (for the metric topology) and so the limit of weight diagrams takes on a fractal aspect. Consequently one is certainly forced to supplement $\mathscr{T}_{2n+1}$ using similar triangles which become smaller and smaller by factors of $g^{-1}$.

These questions lead us to the following. Observe that the essence of Penrose tiling is that the two triangles involved are self-reproducing up to similarity by factors of the Golden Section. Here we show that this property naturally extends to $\mathscr{T}_m$ for all $m \geq 3$.

8.2.   We start with a generalization of Lemma 6.5. Fix a positive integer $n$. Recall the notation of 3.9 and 7.7. Let $\pi := \{\alpha_1, \ldots, \alpha_{2n}\}$ be the set of simple roots in

type $A_{2n}$. Notice that since now $M = \mathbb{Z}g_0 + \ldots + \mathbb{Z}g_{n-1}$, we should write $s_{\alpha,i}\lambda = \lambda - \alpha^{\vee}(\lambda)_{i-1}g_{i-1}\alpha$ and $s_{\beta,i}\lambda = \lambda - \beta^{\vee}(\lambda)_{i-1}g_{i-1}\beta$.

**Lemma.** *Set*

$$\psi(s_{\alpha,i+1}) = s_{2i+1}, \psi(s_{\beta,i+1}) = s_{2n-2i},$$
$$\psi(g_i\alpha) = \alpha_{2i+1}, \psi(g_i\beta) = \alpha_{2n-2i}, \forall i = 0, 1, \ldots, n-1.$$

*(i) $\psi$ extends to an isomorphism of $W^a$ onto $W(A_{2n})$ of Coxeter groups. In particular*

$$\psi(s_\alpha) = \prod_{i=0}^{n-1} s_{2i+1}, \quad \psi(\beta) = \prod_{i=0}^{n-1} s_{2n-2i}.$$

*Set $\widehat{W} = W^a \xrightarrow{\sim} W(A_4)$.*

*(ii) $\psi$ extends to a $\mathbb{Z}\widehat{W}$ module isomorphism of $\mathbb{Z}[g]\alpha + \mathbb{Z}[g]\beta$ onto $\mathbb{Z}\pi$.*

*Proof.* It suffices to verify that the $s_i\alpha_j : i, j = 1, 2, \ldots, 2n$, satisfy the correct identities. Here we just verify one example of a non-trivial case. One has $s_{2i+1}\alpha_{2i} = \psi(s_{\alpha,i+1}(g_{n-i}\beta))$, whilst

$$s_{\alpha,i+1}(g_{n-i}\beta) = g_{n-i}\beta - (g_{n-i}\alpha^{\vee}(\beta))_i\alpha = g_{n-i}\beta + (gg_{n-i})_i\alpha.$$

Yet by 7.7 one has $gg_{n-i} = g_i + g_{i-1}$ and so the right hand side above equals $g_{n-i}\beta + g_i\alpha$, whose image under $\psi$ is $\alpha_{2i} + \alpha_{2i+1}$, as required.

$\square$

8.3. Retain the above notation. We can only view $\mathbb{Z}[g]$ as a lattice in the complex plane when $2n + 1$ is prime, because by Lemma 7.4 the monic polynomial satisfied by $g$, namely $Q_n$ is irreducible over $\mathbb{Q}$ just when $2n + 1$ is prime. We shall avoid this difficulty as follows.

Recall that by 7.4 the largest solution of the equation $Q_n(g) = 0$ is $g = 2cos\pi/2n + 1$. Let $\psi'$ be the composition of $\psi^{-1}$ with evaluation of $g$ at the above value, which is of course well-defined map of $\mathbb{Z}\pi$ into $\mathbb{R}$ which is injective just when $2n + 1$ is prime.

It is clear that the weight diagram of the defining $2n+1$-dimensional representation of $\mathfrak{sl}(2n+1)$ becomes under $\psi'$ is exactly the regular $(2n+1)$-gon. Choose its highest weight to be the fundamental weight $\varpi_1$ corresponding to $\alpha_1$. Let $v_0$ designate the corresponding point $\psi^{-1}(\varpi_1)$ in the plane. For all $i = 0, 1, \ldots n-1$ join the vertex $v_i := \psi^{-1}(\varpi_1 - \alpha_1 - \alpha_2 - \ldots - \alpha_i)$ to $v_{i+1} := \psi^{-1}(\varpi_1 - \alpha_1 - \alpha_2 - \ldots - \alpha_{i+1})$. This gives what we call a zig-zag triangularization of the regular $(2n + 1)$-gon. Such a triangularization is rather natural from the point of view of representation theory. Let $T_i : i = 1, 2 \ldots, n - 1$ denote the triangle with vertices $\{v_{i-1}, v_i, v_{i+1}\}$. From the above lemma or directly one can easily compute their angles and edge lengths.

Set $p_i = P_i(g)$ and let $\mathbf{P}_{2n+1}$ be the monoid they generate in $\mathbb{C}$. Observe that $T_i$ is related to $T_{2n-i}$ by the parity transformation $T \rightarrowtail T'$, which reverses the cyclic order of the vertices.

**Corollary.** *Take $i \in \{1, 2, \ldots, n-1\}$. The angles (up to a multiple of $\pi$) (resp. edge lengths scaled to one for the sides of the $(2n+1) - gon$) in $T_i$ given in cyclic order starting from $v_i$ (resp. $v_{i-1} - v_i$ are $\{1/2n+1, i/(2n+1), (2n-i-1)/(2n+1)\}$ (resp. $\{p_{i-1}, p_i, p_0\}$.*

  **Remark**. Notice that we may now give the conclusion of Lemma 8.2 the following aesthetically pleasing presentation. Consider the extended Dynkin diagram of $A_{2n}$. We may regard it as a regular $(2n+1)$-gon with vertices labelled by the roots $\alpha_i : i = 0, 1, 2, \ldots, 2n$. Then the distance from $\alpha_0$ to $\alpha_i : i = 1, 2, \ldots, 2n$, is $p_{i-1}$. Comparison with Lemma 8.2 shows that this is exactly the factor which multiplies $\alpha$ or $\beta$ in the image of $\alpha_i$.

8.4.   Unless otherwise specified a weight diagram (in the plane) will mean the image under $\psi'$ of a weight diagram of $\mathfrak{sl}(2n+1)$. A weight triangularization of a weight diagram is then defined to be a triangularization in which the vertices are exactly the images of weights of non-zero weight subspaces. For example the zig-zag triangularization of the regular $(2n+1)$-gon defined above is a weight triangularization of the weight diagram of the defining representation. It is not the only weight triangularization possible and unless $n \leq 2$ other weight triangularizations can lead to a different set of triangles (see Figure 2). Nevertheless since there can be no vertices in the interior of the $(2n+1)$-gon every edge must join two vertices on the boundary and hence must come from a root. In particular every edge length must be some $p_i$. The additional triangles obtained in this fashion (which include all possible isosceles triangles with angles which are a multiple of $\pi/(2n+1)$ can be needed for a general weight triangularization (see Figure 3). In general the set $\mathscr{T}_{2n+1}$ (or simply, $\mathscr{T}$ of all triangles obtained from a weight triangularization of the $(2n+1)$-gon is described by the following easy

**Lemma.** *Suppose $T \in \mathscr{T}_{2n+1}$. Then, up to multiples of $\pi/(2n+1)$, the angles in $T$ are given by an (unordered) partition of $2n+1$ into three non-zero parts. The length of the side opposite to the angle of size $\pi k/(2n+1)$ equals $p_{k-1}$ if $k \leqslant n$ and $p_{2n-k-1}$ if $k \geqslant n$.*

8.5.   Fix $m$ an integer $\geq 3$. Label the vertices of the regular $m$-gon by $\{0, 1, 2, \ldots, m-1\}$ in a clockwise order. Take $i_1, i_2, i_3$ with $0 \leq i_1 < i_2 < i_3 \leq m-1$ and let $T_{i_1, i_2, i_3}$ be the triangle whose vertices form the set $\{i_1, i_2, i_3\}$. We may also write $T$ as $T\{i_2 - i_1, i_3 - i_2, i_1 - i_3\}$, that is through its angle set (omitting the multiple of $\pi/m$). Of course we should not want to distinguish such triangles which can be

transformed into one another by rotation; but it is not so obvious whether we should equate triangles interrelated by parity. Indeed if the triangle were a tile with all angles distinct, then its parity translate could only be obtained by flipping it onto its "undecorated" side ! Thus we shall regard $T\{i, j, k\}, T\{j.k.i\}, T\{k.i.j\}$ as the same triangle T; but write $T' = T\{k, j, i\}$ for the triangle obtained from $T$ through parity. In this convention $T = T'$, if $T$ is an isosceles triangle. Let $\mathscr{T}_m$ denote the set of all such triangles. This extends our previous definition for $m$ odd.

8.6. Let $P$ be a polygon (in the Euclidean plane) and $p$ a non-negative integer. Let $pP$ denote the polygon scaled by a factor of $p$, being the empty set when $p = 0$. Suppose $P, P'$ are polygons with share a side of the same length and which do not overlap when fitted together along this side of common length. Then we denote by $P * P'$ the resulting polygon. One should of course appreciate that this notation does not take into account all possible fittings; but for us additional formalism will not serve any purpose. Again it may often be the case that a tiling will violate this condition (see Figure 10). This is in particular true of the tilings obtained through the construction of 8.11.

8.7. Retain the notation and conventions of 8.5. Extending 8.3 we set $T_i = T\{1, i, m - i - 1\} \in \mathscr{F}_m$. Let $p_{j-1} : j = 1, 2, \ldots, m - 1$ denote the distance from the vertex 0 to the vertex $j$. It is the length of a side opposite an angle of size $j\pi/m$ in any element of $\mathscr{T}_m$. Scale the elements of $\mathscr{T}_m$, so that $p_0 = 1$ and set $p_m = 0$. Observe that $p_1 = 2\cos\pi/m$ and that $p_{j-1} = p_{m-1-j}$. Let $\mathbf{P}_m$ denote the monoid generated by the $p_i : i = 1, 2, \ldots, m - 2$.

**Lemma.** *For all $i \in \{1, 2, \ldots, m - 3\}$, one has $T_i * T_{i+1} = p_i T_1$. Moreover $p_1 p_i = p_{i+1} + p_{i-1}$.*

*Proof.* The first part follows by joining the triangles along their common side of length $p_0$. Through similarity of $p_i T_1$ with $T_1$, it implies the second part.

$\square$

**Remark**. Thus $p_i$ is the value of the $(i + 1)^{th}$ Chebyshev polynomial $P_i$ at $x = 2\cos\pi/m$. Via $(*)$ of 2.2, we further deduce that $x$ is the largest (real) root of the equation $P_{[m/2]} = P_{[(m-3)/2]}$.

**Example 1.** Let $T$ be the equilateral triangle of side 1. It is exactly the weight diagram of the defining representation of $\mathfrak{sl}(3)$. Then $T * T * T * T = 2T$ which is a weight triangularization of the six-dimensional representation of $\mathfrak{sl}(3)$. Moreover as is well-known) a weight triangularization of a weight diagram for $\mathfrak{sl}(3)$ can be given in the form $T^{*n}$ for some positive integer $n$. The resulting tiling of the plane goes back to ancient times.

We remark that the relation $T * T * T * T = 2T$ holds for any triangle $T$. In this it suffices match up edges of the same length for then the angles take care of themselves. It results in a tiling of the plane cut out by three infinite sets of parallel lines. Surprisingly it is hardly ever seen or used - perhaps for only technical reasons !

**Example 2.** Consider the pair $T_1, T_2$ of triangles given by 8.3. Their sides have lengths $\{1, g, 1\}$ and $\{g, g, 1\}$ respectively, where g is the Golden Section. Their areas $a_1, a_2$ satisfy $a_2/a_1 = g$. We call it the Golden Pair. Notice that $T_1 * T_2 = gT_1$ by the above and in addition that $gT_1 * T_2 = gT_2$. Inductively we may generate the pair $g^n T_1, g^n T_2, \forall n \in \mathbb{N}$ and moreover in $2^m$ possible ways, where m is the number of products. This fact is the basis of one way of presenting Penrose tiling. (To be precise Penrose constructed a "kite" as $T_1 * T_1$ by joining these triangles along their shortest edge and a "dart" as $T_2 * T_2$ by joining these triangles along their longest edge and then considered *certain* tilings obtained from kites and darts. A useful discussion of this may be found in [16]. We have adopted the more prosaic tilings by the Golden pair as a description of Penrose tiling.)

Just the two triangles of the Golden Pair do not suffice to give all weight triangularizations. Thus although all root lengths are all either 1 or $g$, roots becoming collinear give vertices separated by a distance of $g - 1 = g^{-1}$. Indeed if the highest weight equals $\varpi_2 + \varpi_3$ then the all three triangles $\{T_1, g^{-1}T_1, g^{-1}T_2\}$ are needed for a weight triangularization. In general such differences for example forces one to smaller and smaller triangles until a weight triangulation begins to resemble a fractal.

**Example 3** Take $n = 3$. In the zig-zag triangularization of the regular 7-gon one may replace the lines joining $v_3, v_4$ and $v_4, v_5$, by lines joining $v_3, v_6$ and $v_2, v_6$ and still obtain a weight triangularization (see Figure 2). This results in a new triangle which we shall denote by $T_0$. It is an isosceles triangle with side lengths $\{p_2, p_1, p_1\}$ and angles (up to a multiple of $\pi/7$) which are $\{2, 2, 3\}$. Moreover if we further replace the line joining $v_2, v_6$ by that joining $v_3, v_4$ we obtain the "relation" $T_0 * T_1 = T_2 * T_3$. One might compare the resulting "algebra" of triangles to the commutative ring $\mathbb{Z}[x_0, x_1, x_2, x_3]/ < x_0 x_1 - x_2 x_3 >$. As is well-known the latter is not freely generated.

Return for the moment to the Golden Pair $T_1, T_2$. We noted in Example 2 that it was possible using $*$ to generate all the triangles in the set $\{pT_1, pT_2, \forall p \in \mathbf{P}_5\}$. Moreover we were able to obtain a weight triangularization of the root diagram of $\mathfrak{sl}(5)$ by using just $\{T_1, T_2\}$ scaled by a factor of $g^{-1}$. Here the situation is more complex. Thus in Figure 3 we illustrate a weight triangularization of the root diagram of $\mathfrak{sl}(7)$. It uses $\{T_0, T_2, T_3\}$ scaled by a factor of $p_2^{-1}$. The first surprise is that we cannot just use the set of triangles coming from a zig-zag triangularization of the

weight diagram of the defining representation, namely the set $\{T_1, T_2, T_2', T_2\}$ with whatever scaling. In addition by spotting unions of triangles in Figure 3 we obtain the following relations

$$T_0 * p_2 T_1 = p_1 T_3, \quad T_2 * p_2 T_1 = p_1 T_2, \quad T_2 * p_1 T_3 = p_2 T_2, \quad p_2 T_2 * T_3 = p_2 T_3,$$

$$p_2 T_2 * p_1 T_3 = p_2 T_0, \quad T_0 * T_0 * T_2 * T_3 = p_1 T_0.$$

(This last relation is more difficult to spot but is illustrated in Figure 4.)

We conclude that all the triangles in the set $\{pT_0, pT_2, pT_3, \forall p \in \mathbf{P}_7\}$ can be generated in complete analogy with the case of the Golden Pair. However a further surprise is that we obtain some extra triangles through the relation $T_0 * T_2 = (p_1/p_2)T_0$. A further fact (though perhaps less of a surprise) is that we cannot obtain $T_1$. This is because if $a_i$ denotes the area of $T_i$ then

$$a_0 = (p_1 + p_2)a_1, \quad a_2 = p_2 a_1, \quad a_0 = (p_1 + p_2)a_1,$$

so the area of $T_1$ is too small. On the other hand by Lemma 8.6 and the above $\{T_i : i = 0, 1, 2, 3\}$ generates $\{pT_0, pT_1, pT_2, pT_7, \forall p \in \mathbf{P}_7\}$. Notice that we also obtain the above relations when a given triangle $T$ is replaced by $T'$. (Here only $T_2$ is affected.) We conclude that $\mathscr{T}_7$ generates the set $\{pT : p \in \mathbf{P}_7, T \in \mathscr{T}_7\}$.

8.8. The result in the Lemma 8.7 may be viewed in another way which makes its generalization to other elements of $\mathscr{T}_m$ immediate. Thus let $T_1 \in \mathscr{T}_m$ be the triangle defined in 8.7. Now join the vertex $0$ to any vertex $i : i \in 2, 3, \ldots, m - 1$. The resulting line cuts $T_1$ into two triangles which are easily seen to be similar to two of those in $\mathscr{T}_m$. Moreover there are $m - 3$ such decompositions which come in pairs related by parity. These decompositions are exactly those given in 8.7. This construction immediately gives the result below. Recall the notation of 8.5.

**Lemma.** *Take $0 < i < j < m$. Then for all $t : 0 < t < i$ one has*

$$p_{j-i+t-1}T_{0,i,j} = p_{j-1}T_{0,t,j-i+t} * p_{j-i-1}T_{0,i-t,j}.$$

*Proof.* Cut the given triangle by the line joining the vertex $t$ to the vertex $j$. Then compute angles and side lengths through 8.5, 8.7.

$\square$

8.9. When $n = 3$, the above lemma gives all the pair decompositions given in Example 3. However it does not give the more tricky decomposition of $p_1 T_0$ into four parts. For $n > 2$, the above lemma is insufficient for our purposes because there are missing $p_s$ factors on the left hand side. This arises whenever $j - i > 1$ or taking into account equivalences (under $W$) all sides of the $T_{0,i,j}$ have length $> 1$. However we can then make what we call an inscribed decomposition of $T_{0,i,j}$. Up to a rotation we can assume $i \leq min\{j, m - j\}$. Then assume that $i > 1$, which is the "bad" case.

**Lemma.** *For all $t : 0 < t < i$ one has*

$$p_{2t-1}T_{0,i,j} = p_{t-1}T_{0,i,j} * p_{t-1}T_{0,i+t,j+t} * p_{t-1}T_{0,j-i+t,j} * p_{t-1}T_{0,i,j-t}.$$

*Proof.* Draw a second triangle with vertices $\{t, i+t, j+t\}$. Join the vertices $\{u_1, u_2, u_3\}$ where respectively the line $(0, i)$ meets $(t, i+t)$, the line $(i, j)$ meets $(i+t, j+t)$ and the line $(j, 0)$ meets $(j+t, t)$. The decomposes our original triangle into four triangles with vertices $\{0, u_1, u_3\}$, $\{u_1, i, u_2\}$, $\{u_2, j, u_3\}$ and the inscribed triangle $\{u_1, u_2, u_3\}$. Compute angles and lengths of edges starting from the periphery. This computation is illustrated in Figure 5.

**N.B.** Notice that $T_{0,i,j}$ intersects its rotated twin at six points and the sides of the inscribed triangle are obtained by joining second successive intersection points. However there are two ways to do this and unless $T_{0,i,j}$ is equilateral only the one specified in the proof works ! Namely one must join the points of intersection of a given line of $T_{0,i,j}$ with the corresponding line of its rotated twin. If $T_{0,i,j}$ is equilateral, the second choice corresponds to making a different rotation.

$\square$

8.10.    Yet we are still missing some $p_s$ factors, when $s$ is even. For $m$ odd the first bad case occurs when $m = 9$, which is also the first case when $m$ is odd and not prime. Indeed $\mathscr{T}_9$ admits just one exceptional triangle whose angles are not coprime (up to the factor $\pi/9$), namely $T\{3, 3, 3\}$. Let $\mathscr{T}_9'$ denote its complement in $\mathscr{T}_9$. Using Lemmas 8.8 and 8.9 one may verify that $\mathscr{T}_9'$ generates the set $pT : p \in \mathbf{P}_9, T \in \mathscr{T}_9'$. This again allows one to obtain higher Penrose tiling using nine triangles with angles which are multiples of $\pi/9$.

In order to fill the above lacuna, consider the missing triangle, namely $p_2T\{3, 3, 3\}$ whose decomposition into elements of $\mathscr{T}_9$ is required in order to show that the set generated by $\mathscr{T}_4$ contains $pT : p \in \mathbf{P}_9, T \in \mathscr{T}_9$. Here we may first recall that to decompose $p_iT\{3, 3, 3\}$ for $i = 1, 3$, we cut this triangle with a second one rotated by an angle of $2\pi/9$. This cuts the original triangle into three triangles and an "internal" hexagon. The hexagon was then cut into four triangles by joining second successive edges in either of the two ways possible. The internal triangle is exactly the "inscribed" triangle. Furthermore each of the three triangles in this second set shares a common edge with one in the first set and may be joined to it. The resulting decomposition of $p_iT\{3, 3, 3\} : i = 1, 2$ is exactly what is described in 8.9.

We shall modify this procedure in two ways. First rotate the second triangle by just $\pi/9$. Secondly cut the internal hexagon into six triangles by joining third successive edges (that is opposite edges). This cuts the original triangle into nine smaller triangles. Verification of angles and using the identity $p_2^2 = p_4 + p_2 + p_0$ we obtain the decompositon

$$p_2T\{3, 3, 3\} = T\{1, 3, 5\}^{*3} * T\{2, 3, 4\}^{*6},$$

illustrated in Figure 6.

One remark that if we rotate the second triangle by $3\pi/9$ then its intersection with the first gives a well-known symbol beloved by some. Moreover decomposition the internal hexagon by joining opposite edges gives the decomposition

$$3T\{3,3,3\} = T\{3,3,3\}^{*9}.$$

This generalizes to give (via Remark 1 of 8.7) the relation

$$nT = T^{*n^2}, \forall T \in \mathscr{T}, n \in \mathbb{N}^+. \tag{$*$}$$

8.11. It comes as somewhat of a surprise that the above construction does not generalize in the obvious fashion for all $n$, though the identity $p_2^2 = p_4 + p_2 + p_0$ is still valid for all $n \geq 3$. In other words in order for example to decompose $p_2T : T \in \mathscr{T}_n$ into nine triangles it is not appropriate to cut it with its twin rotated by $\pi/2n + 1$. Nevertheless there is a way to similarly decompose $p_2T : T \in \mathscr{T}_n$ into nine elements of $T \in \mathscr{T}_n$. This is illustrated in Figure 7 for the case $n = 5$ and $T = T\{3,3,5\}$. It can be viewed as being obtained by cutting $T$ with a second copy and decomposing the internal hexagon as before; but the latter does not have vertices on the same circle.

Finally we have illustrated the general decomposition of $p_2T : T = T\{i,j,k\} \in \mathscr{T}_n; i,j,k \geq 3$ into nine triangles symbolically in Figure 8. It is based on the identity $p_2p_i = p_{i+2} + p_i + p_{i-2}$, which holds for all $i : 2 \leq i \leq m - 4$. It is verified using $p_2 = p_1^2 - 1$ and $p_1p_i = p_{i+1} + p_{i-1}$.

In Figure 8 all nine triangles are drawn for convenience though incorrectly as equilateral triangles. The correct angles are given in each corner (as multiples of $\pi/m$). The reader will easily discern a pattern and verify all the needed relations (which are not entirely trivial - for example the angles opposite the edge shared shared by a pair of triangles must be the same. Moreover at external lines must all be straight ones.)

8.12. Finally we describe the decomposition of $p_tT : T = T\{i,j,k\} \in \mathscr{T}_m; i,j,k \geq t + 1$, into $(t + 1)^2$ triangles in $\mathscr{T}_n$. (We remark that the construction does not specifically require t to be even.) First we need the following preliminary. Recall that $p_i = P_i(g) : i = 0, 1, \ldots, n, g = 2\cos\pi/m$ and that $p_i := p_{m-2-i} : (m - 2) \geq i \geq n - 1$ with $g$ being the largest real solution to the identity $p_{[m/2]} = p_{[(m-3)]}$, namely $2\cos\pi/m$. Recall further that $gp_i = p_{i-1} + p_{i+1} : 0 < i < 2n - 1$.

**Lemma.** *For all $i, t \in \mathbb{N}^+ : t \leq i \leq 2n - t - 1$, one has*

$$p_tp_i = \sum_{j=0}^{t} p_{i+t-2j}.$$

*Proof.* One has

$$\begin{aligned} p_t p_i &= (p_1 p_{t-1} - p_{t-2})p_i, \\ &= p_{t-1}(p_{i+1} + p_{i-1}) - p_{t-2}p_i, \end{aligned}$$

from which the assertion follows by induction on $t$.                    $\square$

8.13.   To describe the decomposition of $p_t T\{i, j, k\}$ it is perhaps best to start with the cases $t = 2$ and $t = 3$. The first has been already been described symbolically in Figure 8. The second case decomposing $p_3 T\{i, j, k\} : i, j, k > 3, i + j + k = 2n + 1$ into 16 triangles in $\mathscr{T}_m$ is similarly described symbolically in Figure 9. From these two cases the reader can easily figure out the general solution for himself. The result is described as follows where we use $\prod$ instead of $*$.

**Proposition.** *For all $m, t \in \mathbb{N}^+$ and $i, j, k > t$ with $i + j + k = m$ one has*

$$p_t T\{i, j, k\} = \prod_{c=0}^{t} \prod_{r=0}^{c} T\{i + c - 2r, j + 2c - r, k - t + c + r\}*$$

$$\prod_{c=1}^{t} \prod_{r=1}^{c} T\{j + t - 2c + r, k - t + c + r - 1, i + c - 2r + 1\}.$$

*Proof.* Let us first explain the notation. To begin with $c$ (resp. $r$) labels columns from the top (resp. rows from the left). In the first column there is just one triangle, namely $T\{i, j + t, k - t\}$. Here the angles are given in clockwise order starting from $i$ at the top. Notice that the (upper) edges of this triangle have lengths $p_{j+t-1}$ and $p_{k-t-1}$ as required by Lemma 8.12.

The triangles appearing in the first product above are exactly those which one vertex (with angle $i + c - 2r$) "above" with the remaining two vertices forming a "horizontal" line. Those to the extreme left (corresponding to $r = 0$) have edges of lengths $p_{j+2c}$ which together form the side of $p_t T\{i, j, k\}$ opposite to the angle of size $j$. The sum of their edges equals $p_t p_{j-1}$ via 8.12. Similarly the sum of the edges of those triangles corresponding to $r = c$ is just $p_t p_{k-1}$, whilst the sum of the edges of these triangles in the last row (corresponding to $c = t$) is just $p_t p_{i-1}$.

The triangles is the second sum are inverted relative to the first. Each share a common edge with a triangle in the first set. One checks that the opposite angle sizes coincide. Finally one checks that any vertex has two (resp. 3,6) edges to it and for which the resulting angle sizes are $\pi i$, $\pi j$ or $\pi k$ (resp. any three in cyclic order sum to $\pi$). This means that the large triangle $p_t T\{i, j, k\}$ is cut into $(t + 1)^2$ triangles in $\mathscr{T}_n$ by $3t$ lines with end points on its edges exactly cutting the latter into the sums described in 8.12.                    $\square$

8.14.   Let $< \mathscr{T}_m >$ denote the set of triangles generated by $\mathscr{T}_m$ through $*$. Combining 8.8, 8.9, 8.13 we obtain the following

**Theorem.** *For all $n > 1$ the set $pT : p \in \mathbf{P}_m, T \in \mathscr{T}_m$ is contained in $< \mathscr{T}_m >$.*

8.15. As we already noted the above inclusion may be strict (Example 3 of 8.7). It may also be possible to combine triangles in a different manner than that described using 8.8, 8.9, 8.13. This already occurs for $m = 9$ as illustrated in Figure 10. Nevertheless we have managed to accomplish the program outlined in the last part of 8.1. Notice that in virtue of $(*)$ of 8.10 we may obtain the stronger conclusion defined by replacing $\mathbf{P}_m$ with $\widehat{\mathbf{P}}_n := \{s\mathbf{P}_m : s \in \mathbb{N}^+\}$.

## 9. FUNDAMENTAL DOMAINS, ALCOVES AND THE AFFINE WEYL GROUP.

9.1. In the notation of 2.2 define $\varpi_\alpha, \varpi_\beta$ to be the fundamental weights in $\mathfrak{h}^*$ given by $\gamma^\vee(\varpi_\zeta) = \delta_{\gamma,\zeta} : \gamma, \zeta \in \{\alpha, \beta\}$. It is immediate that if $\pi - \theta$ is the angle between $\alpha$ and $\beta$ then $\theta$ is the angle between $\varpi_\alpha$ and $\varpi_\beta$. In particular the area between the lines they define is a fundamental domain for the action of $W$ in $\mathfrak{h}^*$. Put another way the lines bordering this domain define reflection planes and the group they generate has this domain as a fundamental domain. Notice that in this some integer multiple of $\theta$ must equal $\pi$.

9.2. Now recall the remark in Example 1 of 8.7 where we described a tiling of the plane by a given triangle $T$. Consider the group $W^{aff}$ generated by the reflection planes defined by the three sides of the triangle. One can ask if the given triangle is a fundamental domain for the action of $W^{aff}$ acting on the plane. It is clear that a necessary condition is that this be true with respect to the subgroup generated by just two reflection planes and the wedge shaped domain they enclose. Now in 9.1 we saw that this means that the angle they define must have some integer multiple equal to $\pi$. Thus we must be able to write the angle set of $T$ in the form $\{i\pi/m, j\pi/m, k\pi/m\}$ where $i, j, k$ divide $m$ and of course sum to $m$. We can of course further assume that the greatest common divisor of $i, j, k$ equals one. One easily checks the well-known fact that this condition has just three solutions $i = j = k = 1$, $i = j = 1, k = 2$, $1 = 1, j = 2, k = 3$ corresponding to the fundamental domains (called alcoves) in types $A_2, B_2, G_2$ under the action of the affine Weyl group $W^{aff}$. Alcoves are not disjoint; yet they intersect only in lower dimension and so we may refer to their providing a decomposition of $\mathfrak{h}^*$ into (essentially) disjoint subsets.

We conclude that the tiling of the plane described in Example 1 of 8.7 has rather rarely the special property described above of being the translate of a fundamental domain.

9.3. The above apparently negative result nevertheless leads to the following question. Recall that for any simple Lie algebra $\mathfrak{g}$ of rank n, the Cartan subspace $\mathfrak{h}$ admits a decomposition into alcoves any one of which is a fundamental domain for the action of the affine Weyl group. We may therefore ask if the decomposition into

alcoves in type $A_{2n}$ leads to an aperiodic tiling of the plane through the map $\psi'$ defined in 8.3.

First we briefly recall how alcoves are obtained. (For more details we refer the reader to [1, Chap VI, Section 2]. Fix a system $\pi$ of simple roots and let $\{\alpha^\vee : \alpha \in \pi\}$ (resp. $\{\varpi_\alpha^\vee : \alpha \in \pi\}$ be the corresponding system of coroots (resp. fundamental coweights). Set $Q^\vee = \mathbb{Z}\pi^\vee$. Let $\alpha_0$ be the highest root. In type $A_n$ this is just $\alpha_1 + \alpha_2 + \ldots + \alpha_n$. By a slight abuse of notation we let $s_{\alpha_0}$ denote the linear bijection on $\mathfrak{h}$ defined by

$$s_{\alpha_0}(h) = h - (\alpha_0(h) - 1)\alpha_0^\vee, \forall h \in \mathfrak{h}.$$

It is the reflection in the hyperplane in $\mathfrak{h}$ defined by $\alpha_0(h) = 1$. The affine Weyl group $W^{aff}$ is the group generated by the Weyl group for $\mathfrak{g}$ and $s_{\alpha_0}$. It may be viewed as the Coxeter group with generating set $s_{\alpha_0}, s_{\alpha_1}, \ldots, s_{\alpha_n}$. It is the semi-direct product $W^{aff} = Q^\vee \ltimes W$. Here one uses the fact that $\alpha_0^\vee$ is the highest short root for the dual root system and one checks that $Q^\vee = \mathbb{Z}W\alpha_0^\vee$.

Let $m_i$ be the coefficient $\alpha_i$ in $\alpha_0$. Then the convex set in $\mathfrak{h}$ with vertex set $\{0, \varpi_1^\vee/m_1, \varpi_2^\vee/m_2, \ldots, \varpi_n^\vee/m_n\}$ is a fundamental domain (alcove) for the action of $W^{aff}$ on $\mathfrak{h}$. Since the $\varpi_i^\vee/m_i : i \in \{1, 2, \ldots, n\}$ are fixed points under $s_{\alpha_0}$, it is transformed under $s_{\alpha_0}$ into the alcove with vertex set $\{\alpha_0^\vee, \varpi_1^\vee/m_1, \varpi_2^\vee/m_2, \ldots, \varpi_n^\vee/m_n\}$. The transformation of a given alcove under the reflections defined by their faces (defined by $n - 1$ vertices are similarly described. In type $A_2$ their translates give the tessilation of the plane by equilateral triangles as described in Example 1 of 8.7.

We remark that if $\mathfrak{g}$ is simply-laced then the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ through the Killing form identifies roots with coroots and weights with coweights. In addition in type $A$ the $m_i$ above are all equal to one. We shall use this identification and simplification in the sequel.

9.4.   In order to study the possible consequences of decomposition into alcoves described in 9.3 above for planar tiling we first describe the images of the fundamental weights for type $A_{2n}$ under the map $\psi'$ of 8.3. Fix $n \in \mathbb{N}^+$ and recall the $p_i : i = 0, 1, \ldots, 2n - 1$ as described in 8.12. Let $P(\pi)$ denote the set of (integer) weights in $\mathfrak{h}^*$ relative to the choice of the set $\pi$ of simple roots. Its image under $\psi^{-1}$ is just the $\mathbb{Z}[x]/ < Q_n(x) >$ module generated by the integer weights relative to $\{\alpha, \beta\}$ denoted in 3.5 by $P$. Its image under $\psi'$ is obtained by further evaluation of $x$ as $g = 2\cos \pi/(2n + 1)$. (This makes a difference only if $2n + 1$ fails to be prime.)

**Lemma.** *For all $i \in \{0, 1, \ldots, n - 1\}$, one has*

$$\psi'(\varpi_{2i+1}) = g_i\varpi_\alpha, \quad \psi'(\varpi_{(2n-2i)}) = g_i\varpi_\beta.$$

*Proof.* Recall by Lemma 8.2 that $s_{2i+1}\psi(\lambda) = \psi(s_{\alpha,i+1}\lambda) : i = 0, 1, \ldots, n-1, \forall \lambda \in P$. Equivalently $g_i\alpha_{2i+1}^\vee(\lambda) = (\alpha^\vee(\psi^{-1}(\lambda)))_i$, for all $\lambda \in P(\pi)$. Similarly $g_i\alpha_{2(n-i)}^\vee(\lambda) =$

$(\beta^\vee(\psi^{-1}(\lambda)))_i$. Hence taking $\lambda = \varpi_{2j+1}$, for $j \in \{0, 1, \ldots, n-1\}$ we obtain

$$(\alpha^\vee(\psi^{-1}(\varpi_{2j+1})))_i = \delta_{i,j}g_i, \quad (\beta^\vee(\psi^{-1}(\varpi_{2j+1})))_i = 0, \forall i, j = 0, 1, \ldots, n-1.$$

This gives the first assertion. Taking $\lambda = \varpi_{2(n-j)}$ gives the second assertion. $\qquad\square$

9.5. The above result already has a pleasing geometric feature worth noting. First we recall the relation between the $g_i$ and the chord lengths $p_i$ in the $(2n+1)$-gon, noted in 7.7. Denote $\psi'(\varpi_i)$ simply as $x_i$, for all $i \in \{1, 2, \ldots, 2n\}$. The image $T_f$ of the fundamental alcove under $\psi'$ is the convex set over $\mathbb{Q}$ of $\{\{0\}, x_i : i = 1, 2, \ldots, 2n\}$. It lies in the wedge enclosed by two semi-infinite lines starting at the origin $\{0\}$ and forming an angle of $\pi/2n + 1$. The triangle $T_i$ with vertex set $\{\{0\}, x_i, x_{i+1}\}$ for $i \in \{1, 2 \ldots, n-1\}$ is one of those obtained by a zig-zag triangularization of the weight diagram of the defining representation. (Moreover taken with its parity translate $T'$ having vertex set $\{\{0\}, x_{2n+1-i}, x_{2n-i}\}$ for $i \in \{1, 2, \ldots, n-1\}$ gives all such triangles.) Since the distance between $\{0\}$ and $x_1$ is $p_0 = 1$ the distance between $\{x_i, x_{i+1}\}$, for $i \in \{0, 1, \ldots, 2n-1\}$ is again 1. Joining the points $x_i, x_{i+1} : i = 1, 2, \ldots, n$ gives a triangularization $T_f$ viewed as an isosceles triangle $T\{1, n, n\}$ in $\mathscr{T}_{2n+1}$ with vertex set $\{\{0\}.x_n, x_{n+1}\}$, by exactly the isosceles triangles $p_{i-1}^{-1}T\{2n - 2i + 1, i, i\} :$ $i = 1, 2, \ldots, n - 1$ in $\mathscr{T}_n$. Joining the points $\{\{0\}, x_{2n+1-i}, x_{2n-i}\} : i = 1, 2, \ldots, n$ gives the parity translated triangularization. This result is a direct generalization of the two possible ways to join the triangles $\{T_1, T_2\}$ in the Golden pair to give $gT_1$. Of course it is also natural (read, tempting) to join in addition the points $x_1, x_{2n+1-i}\} : i = 1, 2, \ldots, n - 1$. These give the triangularizations similar to those which appear in Figures 1,3 of the root diagrams in types $A_4$ and $A_6$ except that they involve weights rather than roots.

Augment the above notation by setting $x_0 = x_{2n+1} = \{0\}$. The above result may be summarized by the following "shoelace"

**Lemma.** *The distance between $x_i, x_{i+1}$ is independent of $i \in \{0, 1, \ldots, 2n\}$. Conversely the vertex set of the image $T_f$ of the fundamental alcove can be obtained from the cone with vertex $\{0\}$ and angle $\pi/2n + 1$ by marking on alternate sides of the cone equidistant points (namely the $x_i : i = 0, 1, 2, \ldots, 2n + 1$) starting (and ending) at $\{0\}$.*

9.6. Recall that $\mathfrak{g} = \mathfrak{sl}(2n + 1)$ and that we are identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. The presentation $W^{aff} = Q \ltimes W$ implies that the map $\psi$ defined in Lemma 8.2 commutes with the action of $W^{aff}$. Moreover we may replace $\mathbb{Z}$ in its conclusion by the field $\mathbb{Q}$ of rational numbers. Now view $\mathfrak{h}_\mathbb{Q}$ as the Cartan subalgebra $\mathfrak{sl}(2n + 1, \mathbb{Q})$. Let $\mathbb{Q}'$ denote the number field $\mathbb{Q}[g]$, where as before $g = 2\cos \pi/(2n + 1)$. Then $\psi'$ is a $\mathbb{Q}W^{aff}$ map of $\mathfrak{h}_\mathbb{Q}$ onto $V_{\mathbb{Q}'} := \mathbb{Q}'\alpha + \mathbb{Q}'\beta$, which is injective if $2n + 1$ is prime. In the latter case the (essentially disjoint) decomposition

of $\mathfrak{h}_{\mathbb{Q}}$ into alcoves gives a corresponding (essentially disjoint) decomposition of $V_{\mathbb{Q}'}$. This is less easy to interpret geometrically as the image of each alcove is the convex set of its extremal points over $\mathbb{Q}$, rather than over $\mathbb{Q}'$. As a consequence the images of the alcove interiors appear to intersect when they are naively drawn on the plane.

9.7.  We have already given a description of the image $T_f$ of the fundamental alcove in 9.5. Towards describing the images of the remaining alcoves in $V_{\mathbb{Q}'}$ we proceed as follows. First retain the conventions of 9.7 and revert to our notation of $\widehat{W}$ for the Weyl group of $\mathfrak{sl}(2n + 1, \mathbb{Q})$. Since the corresponding affine Weyl group is the semidirect product of $\widehat{W}$ with the lattice of roots, it follows from 8.2 that we need only describe the image of the fundamental alcove and its $\widehat{W}$ translates. Indeed the images of the remaining alcoves will simply be translates of the former by $\mathbb{Z}[g]\alpha + Z[g]\beta$. This means in particular that there will be only finitely many image types.

   The image of the weights of the defining representation of $\mathfrak{sl}(2n + 1)$ is just the $(2n+1)$-gon with vertex set $\{W\varpi_1\}$. As is well-known the Grassmannian of the corresponding module generates the remaining fundamental modules (together with two copies of the trivial module). Consequently the set $\mathbf{W} := \{\widehat{W}\varpi_i : i = 1, 2, \ldots, 2n\}$ is exactly the set of all sums (different to zero) of distinct elements of the weights of the defining representation and has cardinality $2^{2n+1} - 2$ (with no multiplicities). Its image under $\psi'$ is a union of non-trivial $W$ orbits and hence has cardinality divisible by $2n + 1$. Thus $2n + 1$ must divide $2^{2n+1} - 2$, if $\psi'$ separates the elements of $\mathbf{W}$. This generally fails unless $2n + 1$ is prime.

   Recall that $W = \langle s_\alpha, s_\beta \rangle$ and for each $w \in W$ set $r_w = \mathbb{Q}'w\alpha$ (resp. $p_w = \mathbb{Q}'w\varpi_\alpha$) and $\mathbf{R}$ (resp. $\mathbf{P}$) their union. It follows from 8.2 that the image under $\psi'$ of the roots of $\mathfrak{sl}(2n + 1)$ lie in $\mathbf{R}$. Thus one would have expected the image of $\mathbf{W}$ to lie in $\mathbf{P}$. This is false ! Rather we have

**Lemma.** *The image under $\psi'$ of $\{\widehat{W}\varpi_j : j = 1, 2, \ldots, 2n\}$ lies in the union of the lines $p_w : w \in W$ if and only if $n = 1, 2$.*

*Proof.* Of course the case $n = 1$ is trivial. The case when $2n + 1$ is prime obtains from a simple numerical criterion. Indeed in this case we can easily calculate the cardinality of the intersection $\psi'(\mathbf{W}) \cap p_e$. We claim that it equals $2^{n+1} - 2$ and hence that the cardinality of $\psi'(\mathbf{W}) \cap \mathbf{P}$ equals $(2^{n+1} - 2)(2n + 1)$. Indeed if we take adjacent sums in the weights of the defining representations using the convention that $\varpi_{-1} = \varpi_{2n+1} = 0$, we obtain the set $S := \{\varpi_{2i+1} - \varpi_{2i-1} : i = 0, 1, \ldots, n\}$ of cardinality $n + 1$ in which the $\varpi_{2i} : i = 1, 2, \ldots, n$, have cancelled out. Clearly all ways of cancelling out these elements in taking sums of distinct weights in the defining representation exactly gives all possible non-trivial sums of distinct elements in $S$, the number of which is $2^{n+1} - 2$.

Now we have seen that the cardinality of $\psi'(\mathbf{W})$ equals $2^{2n+1}-2$. Thus $\psi'(\mathbf{W}) \subset \mathbf{P}$ if and only if $(2^{n+1}-2)(2n+1) = 2^{2n+1}-2$, that is when $2^n+1 = 2n+1$, or $n = 1, 2$.

In the case when $2n+1$ is not prime it suffices to exhibit an element of $\mathbf{W}$ whose image under $\psi'$ lies strictly inside the dominant chamber with respect to $< \alpha, \beta >$. If $n \geq 3$ the element $\varpi_1 - \varpi_2 + \varpi_4$ serves the purpose.                □

9.8.   Though our result in 8.14 rather banalizes aperiodic tiling, the above lemma shows that the pentagonal system based on the Golden Section has a special (or perhaps just simplifying) feature.

Let us describe the images of all alcoves in the pentagonal case. As before it is enough to describe those obtain from the fundamental alcove through the action of $\widehat{W}$, since the remaining ones are then obtained through translation by $\mathbb{Z}[g]\alpha + \mathbb{Z}[g]\beta$, where $g$ is now the Golden Section. Let us use $\mathscr{I}$ to denote the image set, namely $W^a T_f$.

Each $T \in \mathscr{I}$ is the $\mathbb{Q}$ convex hull of its vertex set, so it suffices to determine the latter. Every vertex must lie in $\mathbf{W} \cup \{0\} \subset \mathbf{P}$. Every such vertex set must contain $\{0\}$ which is a $W^a$ fixed point. Moreover $W$ acts on $\mathscr{I}$ with every element having a trivial stabilizer because this is already true for $W^a$. Of course the action of $W$ as opposed to the action of $W^a$ is an isometry. Since $\text{card}(W^a/W) = 12$, we have just 12 vertex sets to describe. The first forms the small regular pentagon $P_s$ of which its remaining four vertices, two lie at distance $g^{-1}$ to the origin, two at distance 1 and all in $\mathbf{W}$. One may remark that ten such small pentagons can be formed as expected.

The second forms the large regular pentagon $P_l$ of which its remaining four vertices, two lie at distance 1 to the origin, two at distance $g$ and all in $\mathbf{W}$.

The third is $T_0 := T_f$ itself and in this we note that it is the apex of this isosceles triangle which lies at $\{0\}$. There are four further isosceles triangles $T_i : i = 1, 2, 3, 4$ in $\mathscr{I}$ depending on which of its four remaining vertices lies at $\{0\}$.

Finally there are five rhombi $R_i : i = 0, 1, 2, 3, 4$. Each of these have three sides of length 1 and one side of length $g$ parallel to one of the sides of length 1. Its fifth vertex lies at the intersection of its two diagonals. The latter is at a distance 1 from two of its adjacent vertices on the boundary and at a distance $g^{-1}$ from the opposite two. Thus as we already know from the injectivity of $\psi'$ this "internal" vertex is not in the $\mathbb{Q}$ convex set of the "external" vertices. Any one of the five elements of its vertex set can be chosen to be $\{0\}$ and we use $R_0$ to designate the rhombus with $\{0\}$ as its internal vertex.

As noted in 9.3 for any alcove $A$ there is a unique element $s$ of the affine Weyl group which fixes four of its vertices. Thus there is a unique alcove, namely $sA$ which shares with $A$ the face defined by the four fixed vertices. (Moreover $s$ is the reflection defined by this face.) It follows that the vertex set of $\psi'(A)$ shares exactly

four elements with $\psi'(sA)$. If $\psi'(A)$ is a large (resp. small) pentagon, then $\psi'(sA)$ must be a rhombus (resp. triangle). If $\psi'(A)$ is rhombus then $\psi'(sA)$ can be a triangle or a large pentagon. If $\psi'(A)$ is a triangle then $\psi'(sA)$ can be a rhombus or a small pentagon. For any alcove $A$ there are just five reflections with the above property (each fixing four of the five vertices). Call them the allowed reflections of $A$. If $s$ an allowed reflection of $A$ and $s'$ an allowed reflection of $sA$ different from $s$, we call $\{s', s\}$ an $A$ compatible pair.

**Lemma.** *Let $A$ be an alcove, and $\{s', s\}$ an $A$ compatible pair. Then $A$ and $s'sA$ share exactly three vertices. Conversely any three vertices of $A$ define an $A$ compatible pair $\{s', s\}$ so that $A$ and $s'sA$ share exactly the given vertices.*

*Proof.* Obviously $A$ and $s'sA$ share at least three vertices and cannot share all five. If they share four vertices then there is a reflection $s''$ such that $s''A = s'sA$ forcing $s'' = s's$, which is impossible since all three are reflections. The converse is obvious.    □

9.9.   Retain the above notation and hypotheses. From the above lemma we deduce that $\psi'(A)$ and $\psi'(s'sA)$ share exactly three vertices. The possible triangles they define each have as vertex set any three vertices of the above four shapes, namely $T, R, P_s, P_l$. From these we obtain a straight line segment defined by two vertices with an additional vertex dividing it in the Golden ratio, a Golden pair coming from $P_s$ and a second Golden pair from $P_l$ inflated by a factor of $g$. The remaining triangles coming from $R$ and $T$ are equivalent to these. Conversely a triangularization of one of these shapes obtained by joining its vertices defines a set of compatible pairs of the given alcove.

   The Golden pair $\{T_1, T_2\}$ coming from the triangularization of the small pentagon has side lengths $\{1, g^{-1}, g^{-1}\}$ and $\{1, 1, g^{-1}\}$ respectively in the above normalization. Of the remaining shapes the triangle can be decomposed into $T_1$ and two copies of $T_2$ by lines joining its vertex set. Such a decomposition of $R$ and $P_l$ into members of this Golden pair is (only) possible if we add extra vertices.

   Our general idea is that covering $\mathfrak{h}^*$ by alcoves coming from a fixed alcove $A$ should translate under a *particular* sequence of elements of the affine Weyl group to give a covering of the plane by the triangles obtained Lemma 9.8. Then aperiodicity can be introduced since different sequences can be chosen. However the simplest interpretation of this procedure will not give a tiling since the resulting triangles will overlap. Worse than this the image under $\psi'$ of $P(\pi)$ is dense in the metric topology giving any such tiling a fractal aspect. We describe a rather ad hoc remedy to this situation in the next section.

9.10. **Aperiodic tiling from alcove packing.** Assume $n = 2$, that is consider the pentagonal case. Recall that we have described the image $T_f$ under $\psi'$ of the fundamental alcove. It is an isosceles triangle with apex at $\{0\}$ with its equal sides

of length the Golden Section $g$ the third side being of length 1. Finally it has two extra vertices on its equal sides at distance 1 from its apex. Joining these vertices and further just one of them to an opposite vertex on the base gives it two possible triangularizations satisfying $T = T_1 * T_2 * T_2$.

To proceed we need the following

**Lemma.** *There exists a subset $\mathscr{A}$ of alcoves such that the set $\psi'(A); A \in \mathscr{A}$ tiles the plane.*

*Proof.* Consider the triangle $T$ with vertices $\{\psi'(0), \psi'(5\varpi_2), \psi'(5\varpi_4)\}$ and its parity translate $T'$ with vertices $\{\psi'(5\varpi_2 + 5\varpi_3), \psi'(5\varpi_2), \psi'(5\varpi_4)\}$. Since $5\varpi_2, 5\varpi_3$ both lie in $\mathbb{Z}\pi$, it follows that $\psi'(\mathbb{Z}\pi)$ translates of $T$ and $T'$ tile the fundamental chamber with respect to $W$. Hence their $W$ translates tile the whole plane. Recalling that the affine Weyl group contains $\mathbb{Z}\pi$ and (the image of) $W$ it remains to show that we can choose a subset $\mathscr{A}_0$ of alcoves whose images tile $T$. We choose $\mathscr{A}_0$ so that the images of its elements are again triangles, twenty-five in all. We must show that all the elements of $\mathscr{A}_0$ are translates of the fundamendal alcove under the affine Weyl group. Here we recall that each of the images has exactly five vertices (coming as explained previously as images of the vertices of the corresponding alcoves). Now we noted in 9.8 that the images of the $W^a$ translates of $T_f$ which are triangles come in five $W$ orbits each determined by which vertex of the triangle lies at $\{0\}$. Thus it remains to show that each of the above twenty-five triangles has at least one vertex (and as it turns out only one) which is a $\mathbb{Z}\pi$ translate of $\{0\}$. This is shown in Figure 11, the vertices in question being labelled by the corresponding element of $P(\pi)$ which in each case the reader will check lies in $\mathbb{Z}\pi$. (This is the only non-trivial and slightly surprising part of the proof.) $\qquad\square$

9.11. **Aperiodic tiling from alcove packing, continued.** It remains to give an (aperiodic) triangularization of each $\psi'(A) : A \in \mathscr{A}$. This we shall do using the two previous lemmas. First starting from the fundamental alcove generate the remaining alcoves in $\mathscr{A}$ by taking a sequence of reflections in the affine Weyl group. Of course we get plenty of other alcoves but these we eventually discard. We can add a predecessor to the fundamental alcove different to its successor. Then every $A \in \mathscr{A}$ admits a predecessor $A_-$ and successor $A_+$ obtained by a single generating reflection $s$ (resp. $s'$) with $s \neq s'$. (Of course $A_-$ and $A_+$ need not and will not belong to $\mathscr{A}$.) By Lemma 9.8, the pair $(s, s')$ determines three vertices $v_1, v_2, v_3$ of $T := \psi'(A)$, which we recall is a triangle. Now as explained above we always join the vertices of $\psi'(A)$ lying on its sides and breaking it into $T_2$ and a small rhombus $R$. In addition there will be just two ways of writing $R$ as $T_1 * T_2$. If $v_1, v_2, v_3$ do not belong to the apex of $T$, then they are three vertices of $R$ which when joined gives the required decomposition. Otherwise we take the four vertices of $\psi'(A) \cap \psi(A_+)$ which now has exactly three

vertices of $R$ which we then join. This gives the required (aperiodic) tiling of the plane by the Golden Pair $T_1, T_2$ where the different tilings correspond to different sequences in the affine Weyl group successively running through the elements of $\mathscr{A}$ and of course some discarded alcoves not in $\mathscr{A}$.

Of course all this is a bit of a swizzle, since in particular rather many alcoves are discarded. Again a main point in the construction is to find a subset of alcoves whose images form a tiling of the plane. We found one example but certainly not all. Nor can do we find all tilings of the plane that can be obtained using the Golden Pair. In particular our construction leads to a $1 : 2$ ratio in the contribution of the Golden Pair $T_1, T_2$, whereas one might prefer to have a $1 : 1$ ratio. The latter could be recovered by a equal weighted tiling of the plane using the images of alcoves which are the triangle and small pentagon. Again we could desist in joining the vertices of each $\psi'(A)$ lying on its sides, which was done in all cases and so with no aperiodicity. Then we would obtain a tiling by $gT_1, T_2$ with a $1 : 1$ ratio.

A challenging problem would be obtain a three dimensional analogue of the above construction. Namely for some simple root system $\pi$, find a $\mathbb{Q}W^{aff}$ linear map from $P(\pi)$ to $\mathbb{Q}^3$ and a subset $\mathscr{A}$ of alcoves such that their images form a packing of $\mathbb{Q}^3$. Then use the possible words in $W^{aff}$ to obtain a multitude of (that is to say aperiodic) packings in $\mathbb{Q}^3$.

## 10. THE EVEN CASE

10.1.   We shall describe the analogue of Lemma 8.2 when $W := < s_\alpha, s_\beta > \cong \mathbb{Z}_n \ltimes \mathbb{Z}_2$ with $n$ even. First we need to extend the factorization described in Lemma 7.3. Recall the Chebyshev polynomials $P_n(x)$ defined in  ref 2.2. Set

$$S_n(x) = P_n(x) - P_{n-2}(x), \forall n \geq 1. \tag{$*$}$$

One finds that

$$S_2(x) = P_2(x) - 1, S_1(x) = P_1(x) = x, \quad S_{n+1}(x) = xS_n(x) - S_{n-1}(x), \forall n \geq 2.$$

It is convenient to set $S_0(x) = 1$.

**Lemma.** *For all $n \geq 1$, one has*
$(i)_n \quad S_nP_{n-1} = P_{2n-1}$,
$(ii)_n \quad S_nP_n = P_{2n} + 1$,
$(iii)_n \quad S_nP_{n+1} = P_{2n+1} + x$,
$(iv)_n \quad S_nP_{n-2} = P_{2n-2} - 1$.

*Proof.* From the above formulae and those in 2.2, 8.12, one easily checks the assertions for $n = 1, 2$. For $n \geq 2$ one checks using the above recurrence relations for $S_n, P_n$, that $(ii)_{n-1}, (iii)_{n-2}$ imply $(i)_n$, that $(i)_n, (iv)_n$ imply $(ii)_n$, that $(i)_n, (ii)_n$ imply $(iii)_n$ and that $(i)_{n-1}, (ii)_{n-2}$ imply $(iv)_n$.                                           $\square$

10.2. The following is the analogue of Lemma 7.4.

**Lemma.** *For all $n \in \mathbb{N}^+$ one has*

*(i) The roots of $S_n(x)$ form the set $\{2\cos(2t-1)\pi/2n : t \in \{1, 2, \ldots, n\}\}$,*

*(ii) Suppose $m$ is odd and divides $n$. then $S_d(x)$ divides $S_n(x)$ with $d = n/m$.*

*(iii) $S_n(x)$ is irreducible over $\mathbb{Q}$ if and only if $n$ is a power of $2$.*

*(iv) Take $n > 1$ and odd. Then $x$ divides $S_n(x)$ and the quotient is irreducible over $\mathbb{Q}$ if and only if $n$ is prime.*

*Proof.* By $(*)$ of 2.2 and $(*)$ of 10.1 one has

$$\sin\theta S_n(2\cos\theta) = \sin(n+1)\theta - \sin(n-1)\theta.$$

The right hand side vanishes for $\theta = (2t-1)\pi/2n : t \in \{1, 2, \ldots, n\}$, whereas $\sin\theta \neq 0$. Hence (i). Then (ii) follows from (i) by comparison of roots. Set $z = e^{i\theta}$ with $\theta = \pi/2n$. By (i) the roots of $S_n(2x)$ are the real parts of $x^{2t-1} : t \in \{1, 2, \ldots, n\}$. Since no odd number can divide a power of $2$ these are all primitive $4n^{th}$ roots of unity. These are then permuted by the Galois group of $\mathbb{Q}[z]$ over $\mathbb{Q}$ and so are their real parts. Therefore they cannot satisfy over $\mathbb{Q}$ a polynomial equation of degree $< n$. Hence (iii). The proof of (iv) is similar. $\square$

10.3. As in 2.2 we consider a $2 \times 2$ Cartan matrix with off-diagonal entries $\alpha^\vee(\beta) = -y, \beta^\vee(\alpha) = -1$, regarding $\{\alpha, \beta\}$ as a simple root system with Weyl group given by $W = \langle s_\alpha, s_\beta \rangle$ with the generators being defined as in 2.1. Set $y = x^2$ with $x = 2\cos\pi/m; m \geq 3$. Previously we had considered the case when $m$ is odd, say $m = 2n+1$ and shown (Lemma 8.2) that this root system together with its augmented Weyl group $W^a$ could be obtained from a root system of type $A_{2n}$. Here we establish the related results when $m$ is even. This divides into two cases depending on whether $m$ is divisible by $4$ or not. This is not surprising since the Cartan matrix is a system of type $B_2$ (resp. $G_2$) if $m = 4$ (resp. $m = 6$).

10.4. Observe that $S_k(x)$ is a polynomial in $y = x^2$ if $k$ is even which we write as $T_k(y)$. Again if $k$ is odd then $S_k(x)$ is divisible by $x$ and $\frac{1}{x}S_k(x)$ is a polynomial in $y$ which we write as $T_k(y)$.

Set $m = 4n$ in 10.3. Take $x = 2\cos\pi/4n$. Then $S_{2n}(x) = 0$ by Lemma 10.2 and indeed is its largest (real) root. Let $\pi := \{\alpha_1, \ldots, \alpha_{2n}\}$ be the set of simple roots in type $B_{2n}$. Set $s_i = s_{\alpha_i} : i = 1, 2, \ldots, 2n$ be the corresponding set of simple reflections in the Weyl group $W(B_{2n})$, with $W = \langle s_\alpha, s_\beta \rangle$ defined as in 10.3.

**Lemma.** *Set*

$$\psi'(\alpha_{2(n-k)}) = T_{2k}(y)\alpha, \quad \psi'(\alpha_{2(n-k)-1}) = T_{2k+1}(y)\beta, \forall k = 0, 1, \ldots, n-1,$$

$$\psi'(\prod_{i=1}^{n} s_{2i}) = s_\alpha, \quad \psi'(\prod_{i=1}^{n} s_{2i-1}) = s_\beta.$$

*Then $\psi'$ extends to a $\mathbb{Z}W$ epimorphism of $\mathbb{Z}\pi$ onto $\mathbb{Z}[y]\alpha + \mathbb{Z}[y]\beta$.*

*Proof.* This is straightforward computation using the relations in 2.2 to apply successive products of $s_\alpha, s_\beta$ to $\mathbb{Z}[y]\alpha + \mathbb{Z}[y]\beta$. Compared to the corresponding products in the left hand side of the second equation above applied to $\mathbb{Z}\pi$ this gives two different ways to compute the left hand side of the first equation above and one checks that both give the right hand side. □

10.5.   By 10.2, the map $\psi'$ of 10.4 is injective if and only if $n$ is a power of 2. However we can make it injective for all $n$ by reinterpreting $\mathbb{Z}[y]$ as the ring generated by $y$ and satisfying exactly the relation $T_{2n}(y) = 0$. Of course this ring has zero divisors if $n$ is not a power of 2 and so cannot be embedded in $\mathbb{R}$.

   Let us adopt the above interpretation of $\mathbb{Z}[y]$ so that $\psi'$ becomes an isomorphism. Then we can recover the generating reflections of $W(B_{2n})$ by the same means as used in type $A_{2n}$. Observe that $M := \mathbb{Z}[y]$ is a free $\mathbb{Z}$ module of rank $n$. However it now has *two* natural bases, namely $\{T_{2(n-i)}\}$ which we shall use to define the $s_{\alpha,i}$ and $\{T_{2(n-i)+1}\}$ which we shall use to define the $s_{\beta,i}$, for $i = 1, 2, \ldots, n$. More precisely $s_{\alpha,i}$ is determined by $(*)$ of 3.8 with $m_i : m \in M$ defined by extending $\mathbb{Z}$ linearly the rule $(T_{2(n-i)})_j = \delta_{i,j} : i, j = 1, 2, \ldots, n$ and similarly $s_{\beta,i}$ is determined by $(*)$ of 3.8 with $m_i : m \in M$ defined by extending $\mathbb{Z}$ linearly the rule $(T_{2(n-i)+1})_j = \delta_{i,j} : i, j = 1, 2, \ldots, n$. (This dichotomy is essentially a result of replacing $x$ by $y$.) Then the augmented Weyl group $W^a$ is defined as before to be the group generated by the $s_{\alpha,i}, s_{\beta,i} : i = 1, 2, \ldots, n$. With these conventions we obtain the following

**Lemma.** *Set*

$$\psi'(s_{2i}) = s_{\alpha,i}, \quad \psi'(s_{2i-1}) = s_{\beta,i}, \forall i = 1, 2, \ldots, n.$$

   *Then*
   *(i) $\psi'$ extends to an isomorphism of $W(B_{2n})$ onto $W^a$.*
   *Denote this common group by $\widehat{W}$.*
   *(ii) $\psi'$ extends to a $\mathbb{Z}\widehat{W}$ module isomorphism of $\mathbb{Z}\pi$ onto $M\alpha + M\beta$.*

*Proof.* For example

$$\alpha^\vee(\alpha_{2i-1})_i = -(x^2 T_{2(n-i)+1})_i = -(T_{2(n-i)+2} + T_{2(n-i)})_i = -1,$$

which gives $s_{\alpha,i}\psi'(\alpha_{2i-1}) = \psi'(\alpha_{2i-1} + \alpha_{2i})$, as required. Similarly for example

$$\beta^\vee(\alpha_{2i})_i = -(x^{-1}x T_{2(n-i)})_i = -(T_{2(n-i)+1} + T_{2(n-i)-1})_i = -1,$$

which gives $s_{\beta,i}\psi'(\alpha_{2i-1}) = \psi'(\alpha_{2i-1} + \alpha_{2i})$, as required. □

10.6.   Set $m = 4n + 2$ in 10.3. Take $x = 2\cos\pi/(4n + 2)$. Then $S_{2n+1}(x) = 0$ by Lemma 10.2 and indeed is its largest (real) root. Let $\pi := \{\alpha_1, \ldots, \alpha_{2n+1}\}$ be the set of simple roots in type $B_{2n+1}$. Let $s_i = s_{\alpha_i} : i = 1, 2, \ldots, 2n + 1$ be the corresponding set of simple reflections in the Weyl group $W(B_{2n+1})$, with $W = <s_\alpha, s_\beta>$ defined as in 10.3.

**Lemma.** *Set*

$$\psi'(\alpha_{2(n-k)+1}) = T_{2k}(y)\alpha, \forall k = 0, \ldots, n, \quad \psi'(\alpha_{2(n-k)}) = T_{2k+1}(y)\beta, \forall k = 0, \ldots, n - 1,$$

$$\psi'(\prod_{i=1}^{n+1} s_{2i-1}) = s_\alpha, \quad \psi'(\prod_{i=1}^{n} s_{2i}) = s_\beta.$$

*Then $\psi'$ extends to a $\mathbb{Z}W$ epimorphism of $\mathbb{Z}\pi$ onto $\mathbb{Z}[y]\alpha + \mathbb{Z}[y]\beta$.*

*Proof.* Similar to that of Lemma 10.4.                                  □

10.7.   Take $n = 1$ in 10.6. Then $T_3(y) = y - 3 = 0$ and so $\alpha_3 = \alpha$ and $\alpha_1 = T_2\alpha = (y - 2)\alpha = \alpha$. Consequently $\psi'$ is not injective in this case. Indeed $<\alpha, \beta>$ is a system of type $G_2$ whilst $\pi$ is of type $B_3$. More generally $\psi'$ is never injective and this remains true even if we interpret $M := \mathbb{Z}[y]$ as the ring generated by $y$ satisfying the relation $T_{2n+1}(y) = 0$. The trouble is that $T_{2n+1}$ is a polynomial of degree $n$ in $y$, whilst there are $n + 1$ simple roots which $\psi'$ maps to $M\alpha$. To recover injectivity we define $M_\alpha$ as the ring generated by $y$ satisfying the relation $yT_{2n+1}(y) = 0$, whilst we define $M_\beta$ as the ring generated by $y$ satisfying the relation $T_{2n+1}(y) = 0$. One checks that the conclusion of Lemma 10.6 remains valid when the target space of $\psi'$ is replaced by $M_\alpha\alpha + M_\beta\beta$. (This is false if we also try to make $M_\beta$ to be the ring generated by $y$ satisfying the relation $yT_{2n+1}(y) = 0$.) By construction $\psi'$ becomes injective.

Now determine $s_{\alpha,i}$ by $(*)$ of 3.8 with $m_i : m \in M_\alpha$ defined by extending $\mathbb{Z}$ linearly the rule $(T_{2(n+1-i)})_j = \delta_{i,j} : i, j = 1, 2, \ldots, n + 1$ and similarly determine $s_{\beta,i}$ by $(*)$ of 3.8 with $m_i : m \in M_\beta$ defined by extending $\mathbb{Z}$ linearly the rule $(T_{2(n-i)+1})_j = \delta_{i,j} : i, j = 1, 2, \ldots, n$.

**Lemma.** *Set*

$$\psi'(s_{2i-1}) = s_{\alpha,i}, \forall i = 1, 2, \ldots, n + 1, \quad \psi'(s_{2i}) = s_{\beta,i}, \forall i = 1, 2, \ldots, n.$$

*Then*
*(i) $\psi'$ extends to an isomorphism of $W(B_{2n+1})$ onto $W^a$.*
*Denote this common group by $\widehat{W}$.*
*(ii) $\psi'$ extends to a $\mathbb{Z}\widehat{W}$ module isomorphism of $\mathbb{Z}\pi$ onto $M_\alpha\alpha + M_\beta\beta$.*

*Proof.* For example

$$\alpha^\vee(\alpha_{2i})_i = -(x^2 T_{2(n-i)+1})_i = -(T_{2(n-i)+2} + T_{2(n-i)})_i = -1,$$

which gives $s_{\alpha,i}\psi'(\alpha_{2i}) = \psi'(\alpha_{2i-1} + \alpha_{2i})$, as required. Similarly for example

$$\beta^\vee(\alpha_{2i+1})_i = -(x^{-1}x T_{2(n-i)})_i = -(T_{2(n-i)+1} + T_{2(n-i)-1})_i = -1,$$

which gives $s_{\beta,i}\psi'(\alpha_{2i-1}) = \psi'(\alpha_{2i-1} + \alpha_{2i})$, as required.               $\square$

10.8.   We have constructed the extended Weyl group $W^a$ from $W := < s_\alpha, s_\beta > \cong$ $\mathbb{Z}_m \ltimes \mathbb{Z}_2$ when $m = 4n$ and (resp. when $m = 4n + 2$) and shown it to be isomorphic to $W(B_{2n})$ (resp. $W(B_{2n+1})$. However the construction is rather ad hoc and it is not so obvious what one should do for an arbitrary finite reflection group. Form this construction we may go on to describe the crystals which result in a manner analogous to the case described in 7.7. However this does not seem to be particularly interesting. Yet we note that there are now two ways of interpreting the $B(\infty)$ crystal for $m = 6$, either as a in type $G_2$ or type $D_3$. However these cannot give the same crystal as the number of positive roots is different in the two cases. This is ultimately a consequence of the failure of the injectivity of $\psi'$, which we only rather artificially restored. Again we note the image of the root diagram of $B_3$ under $\psi'$ is just the root diagram of $G_2$ where for example $\alpha_1$ and $\alpha_3$ coalesce to a single root.

    In Figure 12 we have drawn the images under $\psi'$ of the root diagrams in types $B_4$ given a weight triangularization. We recall that in this case $\psi'$ is injective.
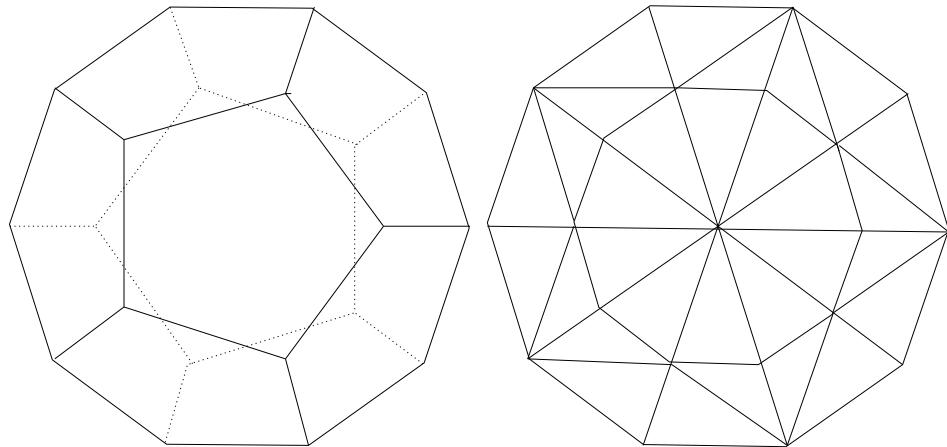
FIGURE 1.

On the left the dodecahedron with co-ordinates given in 6.6. Projected onto one face it gives the pentagonal root system for which a crystal in the sense of Kashiwara is constructed. On the right the root diagram of $A_4$ presented on the plane through the map defined in 6.5 and with a weight triangularization exhibiting a tiling by the Golden Pair.
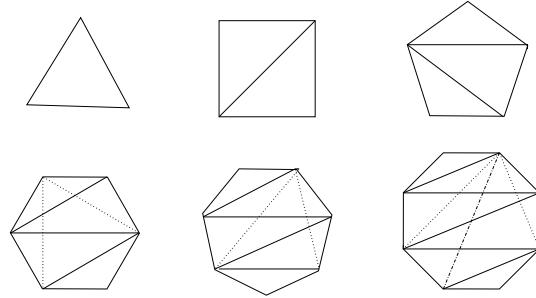
FIGURE 2.

Zig-zag triangularization of regular $n$-gons. For $n = 3, 4, 6$, alcoves are obtained (see 9.2). For $n = 5$ one obtains the Golden Pair 8.7. For $n \geq 6$ additional triangles may be obtained as indicated by the dotted lines. These may be required for further weight triangularizations. For example see Figure 3.
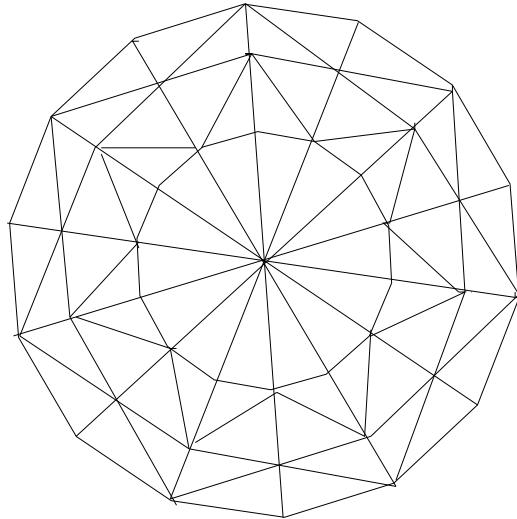
FIGURE 3.

The root diagram given a weight triangularization in type $A_6$ presented on the plane through the map $\psi'$ defined in 8.3.
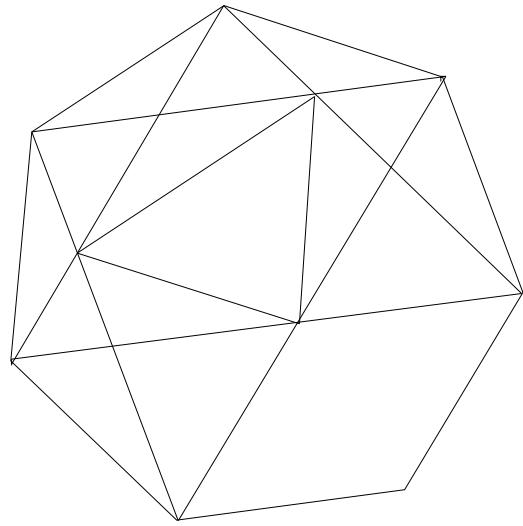
FIGURE 4.

The relation $T_0 * T_0 * T_2 * T_3 = p_1 T_0$. Either one of the triangles of type $T_0$ with vertices on the regular heptagon is cut into four triangles through its intersection with the second such triangle. The resulting four triangles are given by the left hand side above.
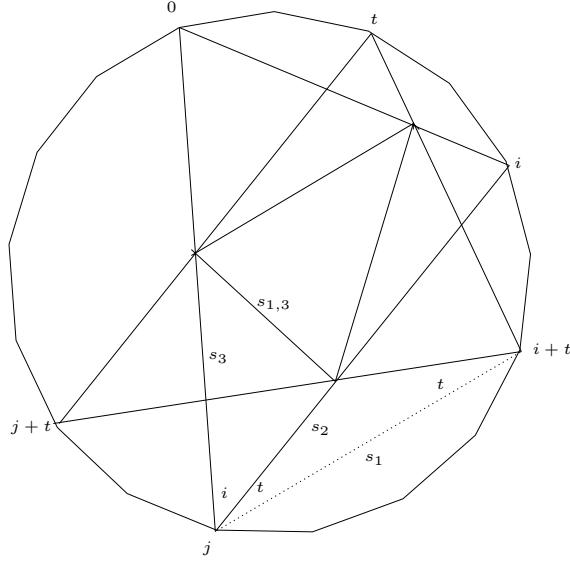
FIGURE 5.

Symbolic presentation of the computation

$$p_{2t-1}T_{0,i,j} = p_{t-1}T_{0,i,j} * p_{t-1}T_{0,i+t,j+t} * p_{t-1}T_{0,j-i+t,j} * p_{t-1}T_{0,i,j-t}.$$

Angle sizes are given up to multiples of $\pi/(2n+1)$, having being computed through the indices of the vertices on the circumference. Consider the isosceles triangle $T := T\{t,t,2n+1-2t\}$ in the lower left hand corner. Its dotted edge has length $s_1 = p_{j-i-t-1}$ because it joins the vertices $i+1,j$. Through $T$, this forces $s_2 = p_{t-1}p_{j-t-i-1}/p_{2t-1}$. In a similar fashion one shows that $s_3 = p_{t-1}p_{j-t-1}/p_{2t-1}$. The triangle $T'$ with these two edge lengths subtending an angle $i$ is hence completed determined and is given by $T' = T\{i,j-i-t,2n+1-j+t\}$. This in turn implies that $s_{1,3} = p_{i-1}p_{t-1}/p_{2t-1}$. Repeating this computation for the other two sides of the central triangle $T''$ shows it to be $p_{t-1}/p_{2t-1}T_{0,i,j}$. The data for the three remaining triangles which form $T_{0,i,j}$ are simultaneously obtained and together give the required assertion.
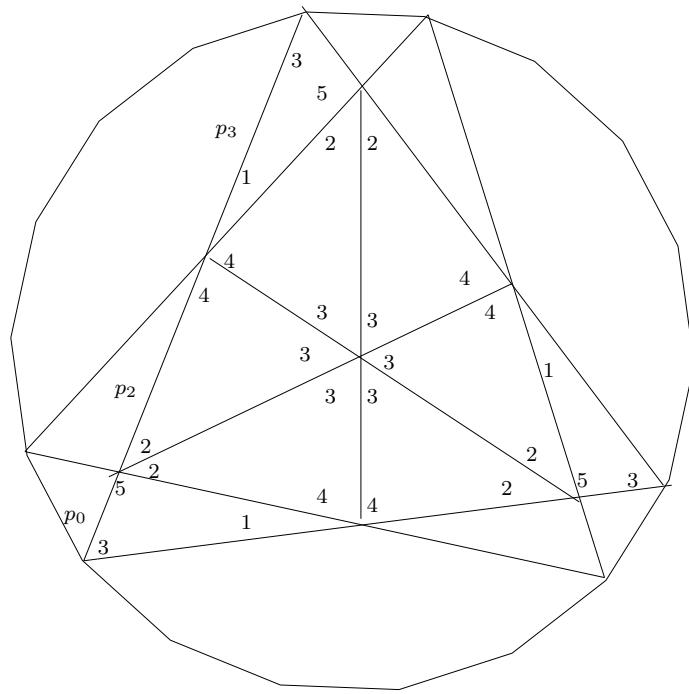
FIGURE 6.

Decomposition of $p_2 T\{3, 3, 3\}$ into nine triangles illustrating $(*)$ of 8.10.
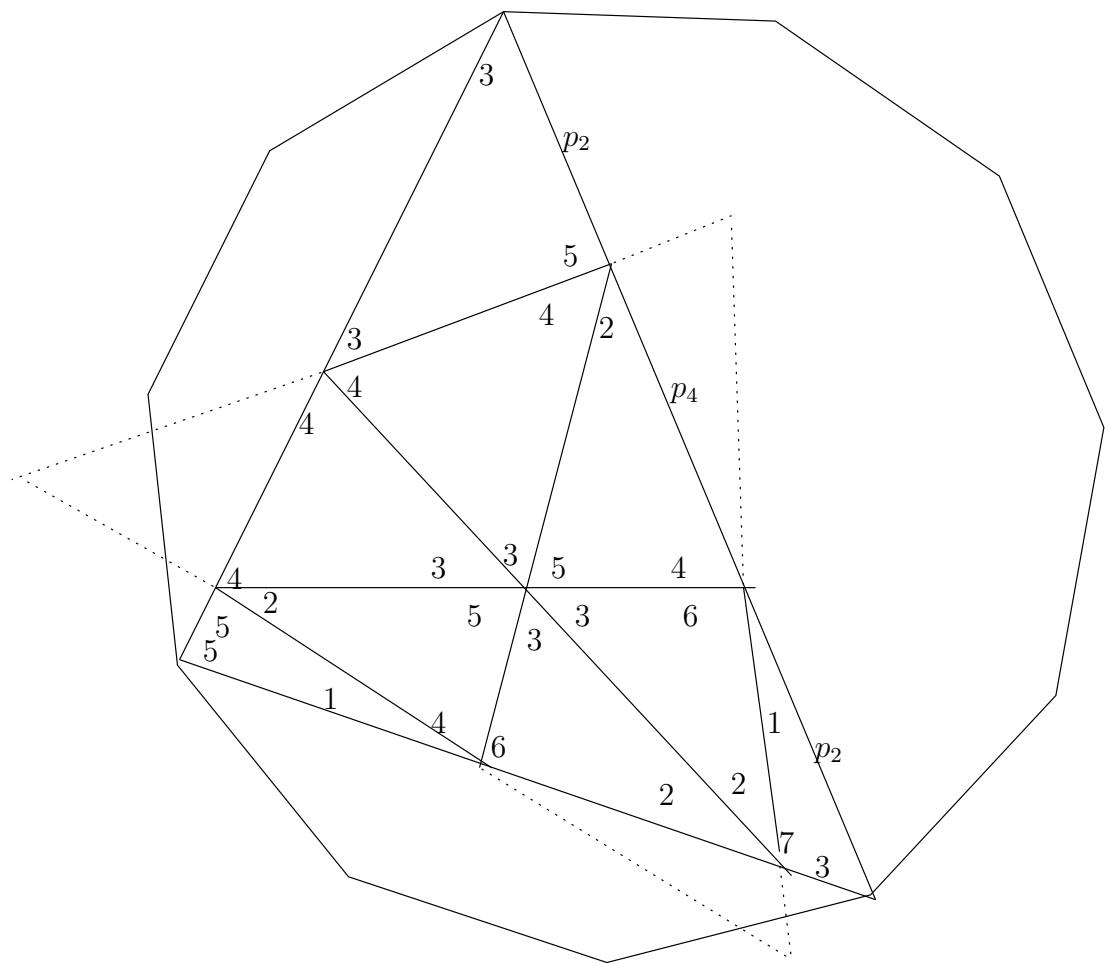
FIGURE 7.

Decomposition of $p_2T\{3, 3, 5\}$ into nine triangles. To be contrasted with the previous figure.
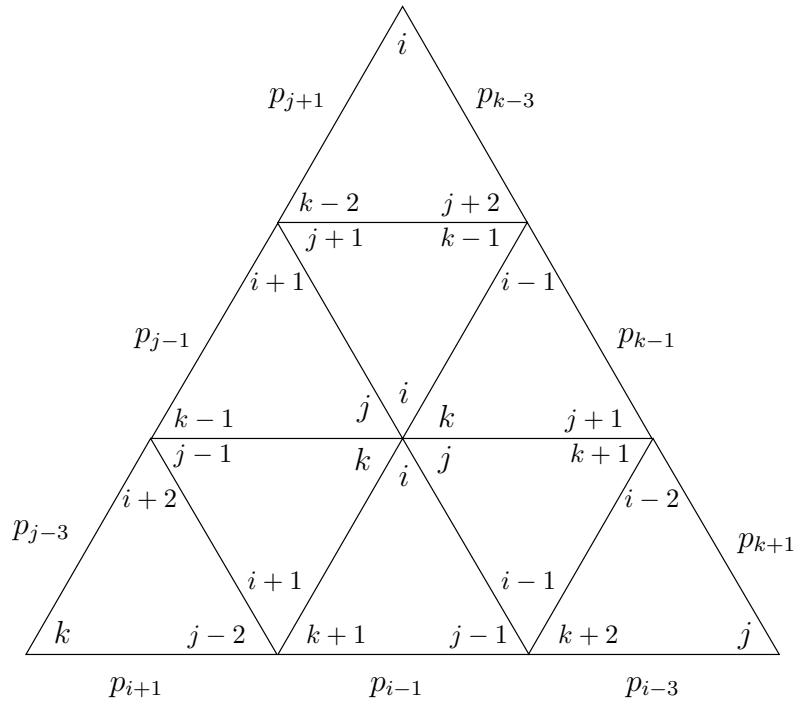
FIGURE 8.

Symbolic presentation of the decomposition of $p_2 T\{i, j, k\}$ into nine triangles in $\mathcal{T}_{2n+1}$, whose angles (as multiples of $\pi/(2n + 1)$) and edge lengths are as indicated. In this $i, j, k \geq 3$ and $i + j + k = 2n + 1$, with $n$ an integer $\geq 4$.
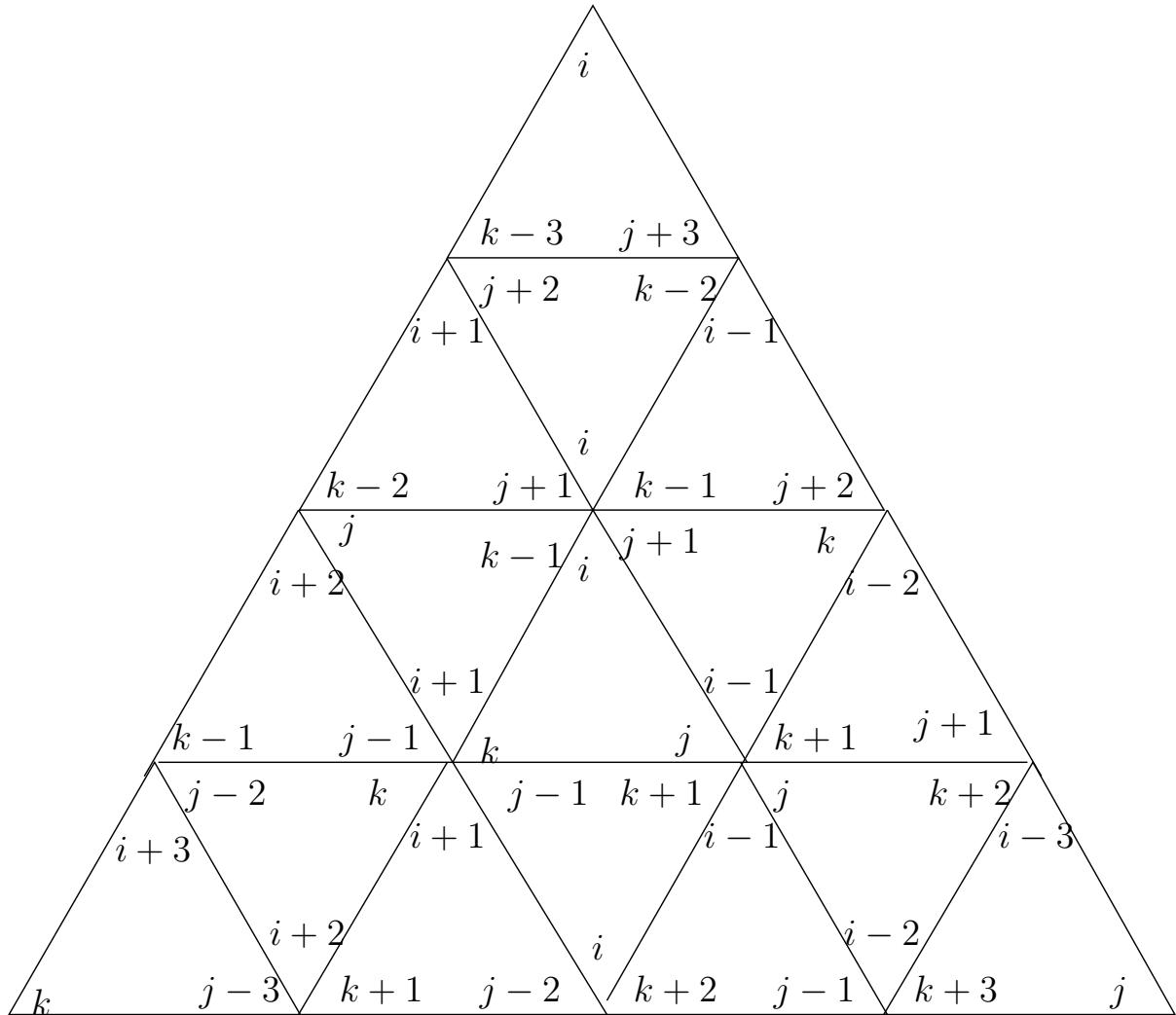
FigURE 9.

Symbolic presentation of the decomposition of $p_3 T\{i, j, k\}$ into 16 triangles with angles as multiples of $\pi/2n + 1$ indicated.
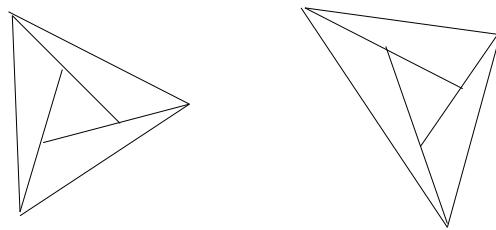
FIGURE 10.

Non-standard decompositions of $T\{3, 3, 3\}$ and $T\{3, 3, 5\}$, that is not satisfying 8.6.
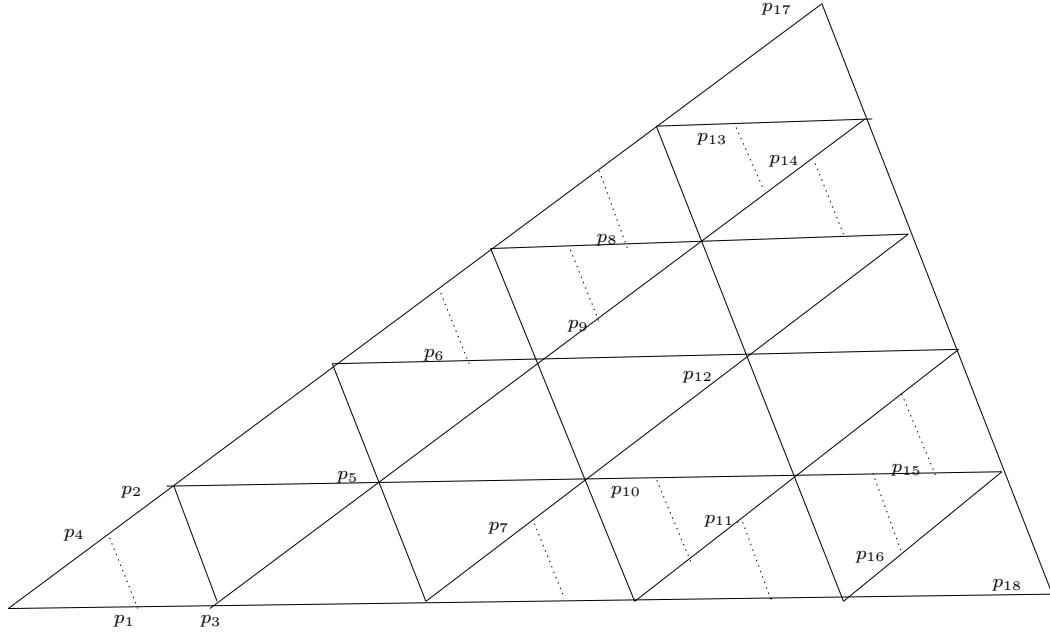
FIGURE 11.

Detailing the last part of Lemma 9.10. One has $p_i = \varpi_i : i = 1, 2, 3, 4$. Then $p_i : i = 5, 6 \ldots, 18$ are computed by vector addition. One checks that all the latter lie in the root lattice. For example $p_5 = \varpi_2 + \varpi_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$ and $p_6 = \varpi_1 + 2\varpi_2 = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$.
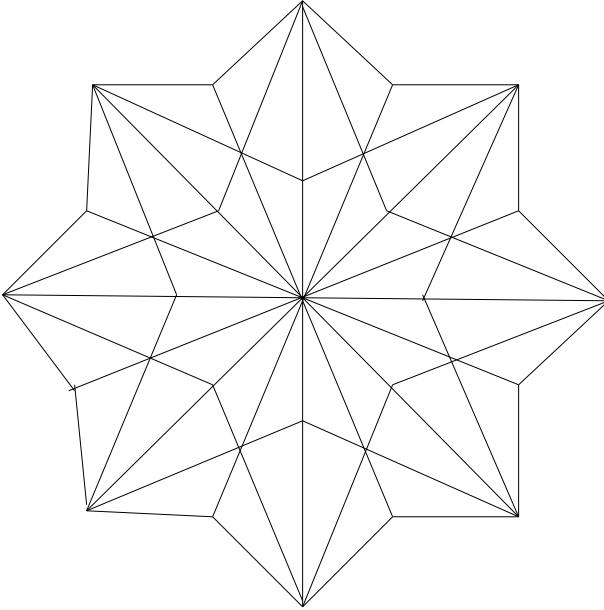
FIGURE 12.

Root diagram in $B_4$ given a weight triangularization.

## Acknowledgements

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